BUILDING BRIDGES TO ALGEBRAIC THINKING

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Abstract

Currently, mathematics researchers and educators argue and describe that algebra is a way of thinking, a method of expressing relationship, describing and representing patterns, and exploring mathematical properties. Thus, 'algebraic thinking' has become a catch-all phrase for the recent research. However, many students still approach algebra as formal structure, manipulation of symbols and rote skills. The SOLO model developed by Biggs and Collis can be adapted to provide a useful four-step template of generalized questions, that lead and help students to build a bridge in making transition from the basic concept of algebra to the advanced skills of representation and generalization of linear pattern by using algebra symbols.

Introduction

The technological future of a modern society depends in large part on the mathematical literary as mathematics is the most powerful technique for the understanding, generalizing of patterns and analysis of the relationship of the patterns. As what was said by Sawyer, mathematics is a search for all patterns and that appear all around us in nature, in science and in human behavior. Thus, students need to be equipped with the skill to represent patterns such as number pattern, pictorial pattern and so on in all branches of mathematics by using variable in expressions, equation, formula, and inequalities.

Algebra is the most important topic to make generalization and interpretation of patterns and relationships. Therefore, it has potential value for many interesting areas. For instance, it is applied to utilize hundreds of variables to monitor the location of aircraft or plan the varied schedules, design electrical circuits and microchips involving millions of transistors (David, 2003). Besides, other main branches of mathematics such as geometry, statistics, and probability need to use the concept of algebra for describing and reasoning about patterns.

Consequently, to fulfill the wide demand of the advancement in the use of algebra in science and technology today, a great reform of curriculum in mathematics education had been done in many countries. Currently, mathematics researchers and educators have challenged the conventional view of algebra as formal structure, manipulation of symbols and rote skills. They argue and describe that algebra is a way of thinking, a method of expressing relationship, describing and representing patterns, and exploring mathematical properties (Day &Jones, 1997; Herbert & Brown, 1997; Moses, 1997; NCTM, 2000; Thornton, 2001). Thus, 'algebraic thinking' has become a catch-all phrase for the recent research.

What is Algebraic Thinking?

Algebraic thinking had been discussed by many educators and researchers. It is hoped that by presenting these ideas, a better understanding of this phrase will prepare students and teachers towards more successful in teaching and learning of algebra in the secondary mathematics education.

Herbert and Brown (1997) concluded that algebraic thinking is using mathematical symbols and tools to analyze different situations by: i) extracting information from the situation. ii) representing that information mathematically in words, diagrams, table, graphs and equation, and iii) interpreting and applying mathematical findings such as solving for unknown, testing conjectures, and identifying functional relationships.

Meanwhile, Kaput (NCTM, 1993) said that algebraic thinking involves the construction and representation of patterns and regularities deliberate generalization and most important, active exploration and conjecture.

Kriegler (2000) viewed that algebraic thinking is organized into two major components: the development of mathematical thinking tool and the study of fundamental algebraic ideas. Mathematical thinking tools include analytical habit of minds especially problem solving skills, reasoning skills and representation skill. Meanwhile the fundamental algebraic ideas represent a domain in which algebraic thinking tools can develop. That is the content, the 'meat' for the subject study such as properties of number system, operation sense and symbol manipulation (variable, expressions, equation, formula and inequalities). These two components are related. Therefore, one can hardly imagine thinking algebraically with nothing to think about algebraic ideas or otherwise.

Based on the NCTM standards (Grade 5-8) (NCTM, 1989), algebraic thinking is about understanding of concept variable, representation skill, exploration of interrelationships of representation and the analysis of the representation to find the relationship, solving equation, investigate and apply algebraic method to solve a variety of real- world problems and mathematical problems.

Most of the definitions on algebraic thinking are based on identifying the kinds of algebraic processes in solving algebra tasks or process-based (Herbert & Brown, 1997; Kaput, 1993; NCTM, 1989). Generally, there is no single definition of algebraic thinking. It can be viewed from different perspectives. A mathematician's view of algebraic thinking is not usually the same as the view of a psychologist, an elementary school teacher or an expert on algebraic thinking. However, we provide some perspectives of the nature of algebraic thinking if the question is rephrased: What kinds of algebraic processes demonstrate ability in algebra? Friedlander & Hershkowitz (1997), Herbert & Brown (1997) and Swafford & Langrall (2000) maintained that the ability of using equations to make generalization for the pattern involves a number of algebraic process or phases: investigating the pattern, representing and generalizing the pattern, interpreting and applying the equation.

Concerns and Research on the Development of Algebraic Thinking

According to Martinez (2003), many students still approach algebra as an ‘alien’ domain with an incomprehensible language. According to Martinez (2003), algebra students often make complaints like these:

> 'I like numbers, but these x's, y's and z's are confusing. They don't mean anything, and mathematics is supposed to mean something.'
>
> 'Mathematics is supposed to be about numbers, letters are for English class.'
>
> 'I like mathematics. I can use. But I'll never use algebra to solve problem.'
>
> 'Algebra is the hardest mathematics because of all the x's and **y**'s.

Algebra conjures classroom memories of x's and y's, manipulating symbols, numbers, variables and solving the unknown for the equation or alternatively finding output for the specific inputs rather than conceptual understanding and mathematical thinking application in solving problem situation. The comments above pointed to the stereotypical image of algebra classes as being unpleasant and boring. That is, most people think algebra as a discipline involving the manipulation of symbols and rote skills. It acts more like a 'wall', presenting an obstacle that they find too difficult to cross and yet many of them are unable to connect algebra with real life or to view it as a necessary and useful tool in their daily lives. It is seen as the only mathematics course required.

According to Steen (1992), for most students, the current school approach to algebra is still an absolute disaster in United States. Half of the students who take the first year algebra leave the course with a lifelong distaste for mathematics. Linda and Sara (2000) found that a failure in the college-preparatory algebra course rate of 40-50 percent is typical.

In Malaysia, a great deal of research has been done in recent years to investigate the variety of misconception in algebra (Cheah & Malone, 1996; Teng, 2001, Lim, 2007). Research studies (Teng, 2002; Tall and Razali, 1993) have noted that symbol manipulation and procedural skills practice in algebra class among the secondary school students might serve to prolong the interpretation that algebra is a 'menagerie' of disconnected rules to deal with different contexts ('collect together like terms', 'turn upside down and multiply', 'do the same thing to both sides', 'change side, change sign' etc). It therefore exhibits the poor understanding of the basic concept and cognitive obstacles among the students as this practice to algebra relies almost exclusively on written symbolic forms as the tool to make representation, generalization and interpretation to the applied problem. Salmah and Zailah (1999) revealed that the failure of our students to perform well in algebra is due to the poor basic understanding of concept and they always feel that it is a difficult subject.

All these problematic situations lead the interest of researchers to adapt the SOLO (Structured of the Observed Learning Outcome) model to provide a useful four-step template of generalized questions, that lead and help students to build a bridge in making transition from the basic concept of algebra to the advanced skills of representation and generalization of pattern.

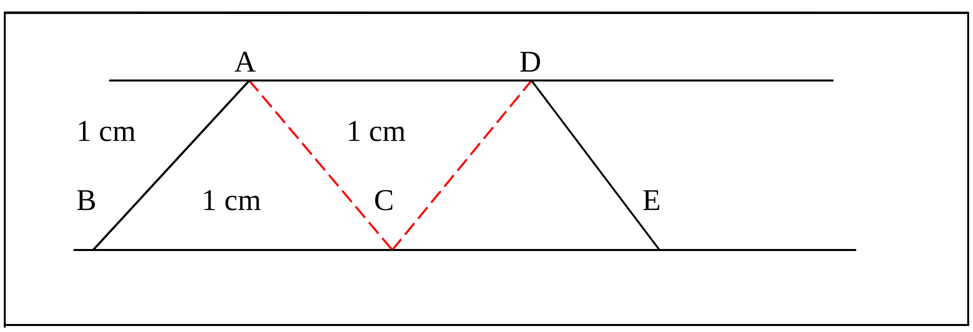
The SOLO taxonomy was developed by Biggs & Collis (1982). It is designed mainly as a mean to investigate students' cognitive ability in school learning context (Biggs & Collis, 1982; Collis & Romberg, 1997; Chick, 1988; Collis, Romberg & Jurdak, 1986). It had been used to analyze the structure response of student's problem solving ability, mathematical thinking ability and understanding of mathematical concepts over a wide educational span from primary to tertiary levels. It also had been applied in the area of science, counseling and practice subject.

SOLO model provides a framework to classify the quality of response which can be inferred from the structure of the answer to a stimulus. When students answer the tasks given, their responses to the task can be summarized in terms of five levels (Biggs & Collis, 1982; Biggs & Collis, 1989), ranging from prestructural to extended abstract; these are described below:

a. Prestructural - the learner does not understand the point of question. He/She fails to engage in the problem.
b. Unistructural - the learner focuses on one or a few relevant information given to give a response to the direct concrete reality involved in the problem. The information is obtainable from either the stem or from the diagram given.
c. Multistructural - the learner picks up more relevant information given to obtain the solution but does not integrate them. The information given may use as a recipe where a set of instructions are followed in sequence to solve the problem.
d. Relational - in this level, the learner integrates all aspects of given information with each other into a coherent structure.
e. Extended abstract - the learner generalizes the structure into a new and more abstract situation. This may allow generalization to a new topic or area.

Since the SOLO model is based on the analysis of material presented, rather than the individual presenting the material, it has the potential to be used as a powerful teaching tool. The following discussion outlines an adaption of this model that provided the Form Two students and their teachers with a pedagogically sound template which can be used to develop students' algebraic thinking in making generalization of linear pictorial pattern.

Problem situation: Triangle Train
Look at the triangle train below. The **length** of triangle train is determined by **the number of** equilateral triangles with the side 1cm. The perimeter of the triangle train is 5 cm (AB, AD, DE, BC, and CE) if the length (the number of triangle) is 3 (**interior lines don't count as part of perimeter**).



a. Unistructural competence: finding the next term

What is the perimeter of the triangle train if the length is 4?

Descriptors:

At this level teachers should encourage students to comprehend the linear pattern. Thus the question only requires the response based on referring the concrete information (given terms in the diagram) to find the next term for the given sequence. Normally, this question can be answered most simply by drawing and counting.

Example of Form Two Students' Responses
Majority of the students were able to solve and respond correctly by seeking out the sequence of pattern which comes next to it or extend the sequence by referring directly to the diagram and information given.

The following dialogues depicted how Lina responded to this question:

Teacher: Please tell me the perimeter of the triangle train if the length is 4?
Lina: (points and counts the perimeter of the train). 1, 2, 3,…,5, 6 cm.

b. Multistructural competence: finding the lower specific cases
What is the perimeter of the triangle train for the length (number of train) of 6, 10 and 15?

Descriptors:

At this level, the work under examination shows evidence of using the information given serially, without forming the structure to generalize it. The questions require the given information are handled serially. That is, identify the recursive relationship between the terms in the sequence in order to compute some lower specific cases.

Example of Form Two Students' Responses
Most of the students were able to notice the linear pattern in the diagram. They solved the questions by counting method. They drew the trains to count the perimeter. Figure 1 shows Nick's response and figure 2 shows Hafiz's response.

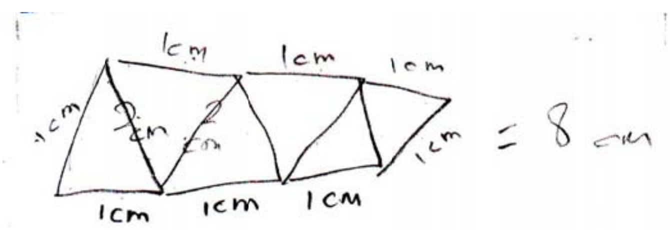


Figure 1 Nick's response for multistructural level question

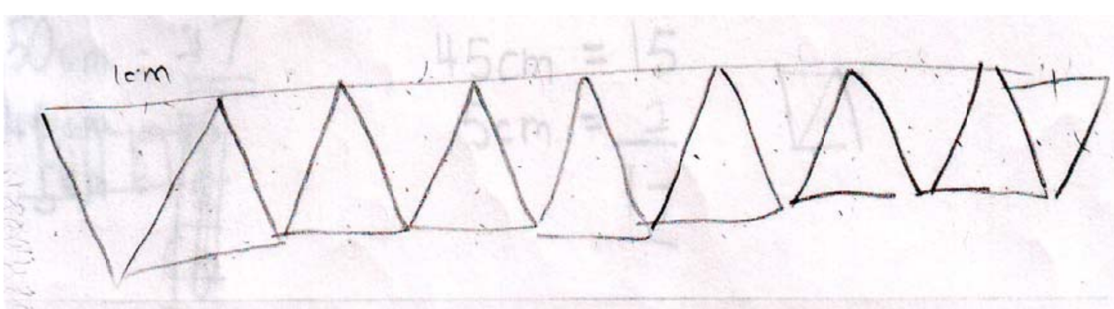


Figure 2 Hafiz's response for multistructural level question

However, John, Hafiz and Lina tried to identify the linear relationship involving an arithmetic operation. They did not use the manipulative to get the solution instead of substituting the specific values into the arithmetic expression that they formed. These were shown in the extracts below:

Teacher:	If the length of the triangle train is 15, what is the perimeter?
John:	(pause) If I say 17 cm
Teacher:	How you get 17 cm?
John: first	1 triangle times one because 1 triangle only have 1 cm if I cover the part of the first triangle first. Here have 15 triangles, so 15 times 1 then plus 2.
Teacher:	Why you want to plus 2?
John:	Because the side of first triangle have 1 cm and the last part of triangle train is also 1 cm. (Refers the figure 3)

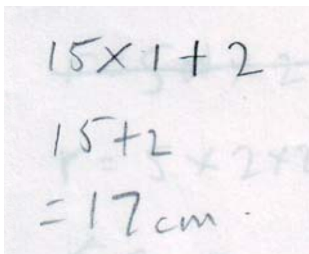


Figure 3 John's response for multistructural level question

c. Relational competence: making generalization and application of the rule

i. What is the perimeter of the triangle train for the length (number of train) of 50, 80, 120?
ii. What is the perimeter of the triangle train for the length (number of train) of *h*?
iii. Try to write a linear equation to find the perimeter of the triangle train for any length of the triangle train. Let *r* represents perimeter of the train and *s* represents the length of the train.
iv. If the triangle train has a perimeter of 50cm, what is its length? Try to apply the linear equation to solve this problem.

Descriptors:

The most important feature of the relational stage is that the students should be able to inter-relate all given information to make generalization and apply the rule to solve the related situation. If a student provides this response, it would demonstrate his/her algebraic thinking in: i) identifying the linear relationship between the variables. ii) representing the relationship between the two variables in a table or graph. iii) constructing a rule in the form of arithmetic expression, algebraic expression and linear equation. iv) applying the rule in related situation.

Example of Form Two Students' Responses

John, Lina and Hafiz tried to link their interpretation and mathematical findings to connect the counting action with an accurate symbolic representation in the form of arithmetic expression, algebraic expression and linear equation. Then they tried to analyze the problem given into the rule that they formed. Figure 4 shows John's response.

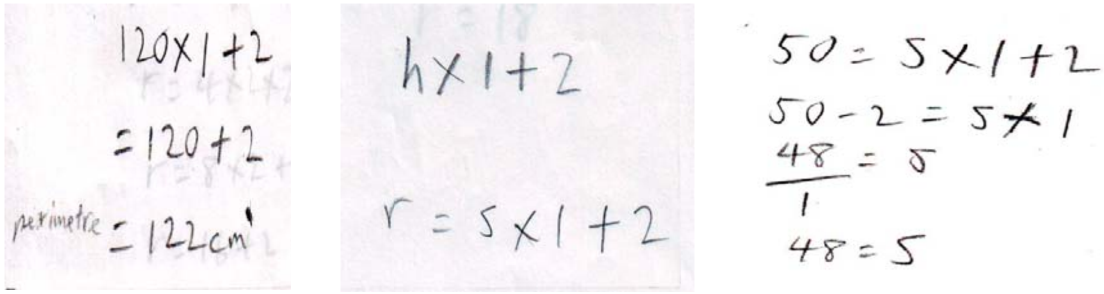


Figure 4 John's response for relational level questions

d. Extended abstract competence: creating new pattern of train

Can you try to suggest a new pattern of the train and form a linear equation to represent the perimeter ***(r)*** of the train for any length (***s***)?

Descriptors:

This level represents the highest level of algebraic thinking that demonstrates critical and creative thinking in the linear pattern. The students are required to show their abilities to extend the application of the given information in the new situation (creating new pattern) and recognize an alternative approach which is formed by the abstract concept (linear relationship).

Example of Form Two Students' Responses

Lina showed her ability to extract the abstract concept (linear pattern) from the information given to form the linear equation. Then, she generalized and represented the square train that she created. These were shown in the extracts below:

Teacher: Try to create a new pattern of train?
Lina: I like square train.
Teacher: Can you form a linear equation to represent the perimeter ***(r)*** of the train for any length (***s***)?
Lina: (Thinks and Counts some specific cases). Ya, I got it. (r x 2) + 2 = p. (Refers the figure 5)

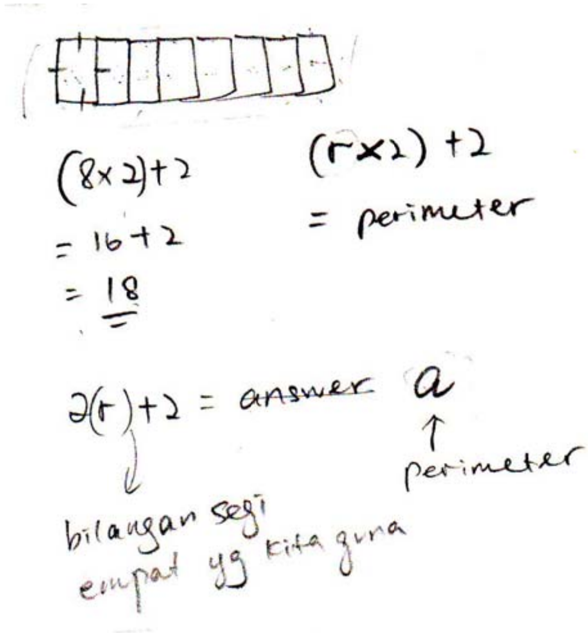


Figure 5 Lina's response for extended abstract level question

Conclusion

The ongoing interactions suggested by these questions enable teachers to capitalize on students' problem-solving skill by using different focus, to extend their thinking beyond specific cases and to generalize their thinking by forming arithmetic rule and algebraic rule. The questioning sequence described uses Biggs' SOLO model allows teachers to promote a better understanding of linear pattern, and focuses the students' attention on to a specific set of features at each level of development. Besides, the above template highlights the general types of questions that served as a bridge to enable teachers guide their students to make the transition from arithmetic thinking to algebraic thinking by using SOLO-based techniques. This pattern can be referred by teachers to provide their own questions especially for other topics in algebra.

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