APPLICATION OF MATHEMATICS PROFICIENCY MODEL IN TEST DEVELOPMENT

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Abstract

Mathematics is widely regarded as one of the most important subjects in the school curriculum. Indeed, it is likely that more lessons of mathematics are taught in schools throughout the world than any other subject. When concern is expressed about the performance of pupils, mathematics is usually singled out as being a particularly worrying problem. It seems that the whole world regards it as important that children should be able to demonstrate a high level of proficiency in the subject. The purpose of this conceptual paper is to explain the mathematics proficiency model as proposed by Kilpatrick, Swafford, and Findell (2001) that consists of five strands, namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Furthermore, this article discusses how these strands and their respective characteristics can be used as a foundation to develop the Form Two mathematics test which is closely aligned with the Malaysian mathematics curriculum specification. Examples of the test item representing each strand and combination of strands are also presented.

Introduction

Since the publication of Bloom’s taxonomy in 1956, psychological and educational research has witnessed the introduction of several theories and approaches to learning which make students more knowledgeable of and responsible for their own learning, cognition, and thinking (e.g. constructivism, metacognition and self-regulated learning). In addition, the use of Bloom’s taxonomy in test development is not uncommon among test developers. It included six major categories, namely knowledge, comprehension, application, analysis, synthesis and evaluation. It was intended to provide for classification of educational system goals, especially to help teachers, administrators, professional specialists, and research workers to discuss curricular and evaluation problems with greater precision (Bloom, 1994, p.10). One of the most frequent uses of Bloom’s taxonomy has been to classify curricular objectives and test items in order to show the breadth, lack of breadth, of the objective items across the spectrum of the six categories.

The structure of Bloom’s taxonomy is a cumulative hierarchy: hierarchy because the classes of behaviours are arranged in order of increasing complexity and cumulative because each class of behaviours is presumed to include all the behaviours of the less complex classes. It is assumed that mastery of each simpler category is prerequisite to mastery of the next more complex one (Krathwohl, 2002, p.213).

In the application of the Bloom's taxonomy, several weaknesses and practical limitations have been revealed. A notable weakness is the assumption that cognitive processes are ordered on a single dimension of simple-to-complex behavior. As required in a cumulative hierarchy, the categories were presumed not to overlap. Anderson, Krathwohl, Airasian, Cruikshank, Mayer, Pintrich, Raths and Wittrock (2001) point that the term "cumulative hierarchy" which indicates the mastery of a more complex category requires prior mastery of all the less categories below it is inflexible. However, in applying Bloom's taxonomy, Ormell (1974) reported contradictions in the frequent inversion of various objectives and tasks. For example, certain demands for knowledge are more complex than certain demands for analysis or evaluation. In addition, evaluation is not more complex than synthesis; synthesis involves evaluation (Krietzer & Madaus, 1994).

In the context of mathematics testing, the use of Bloom's taxonomy can be seen as too general in test development. For example, knowledge in Bloom's taxonomy is described as the thinking skill that a student can recall or recognise information, concepts and ideas. This description does not focus on mathematics but rather a general statement on the meaning of knowledge. It is already known that mathematics comprises many important concepts, procedural and analytical skills. As a consequence, a model that focuses on mathematics is therefore suggested to describe these concepts, procedural and analytical skills. One of the models that employed the constructivism theoretical framework is the mathematics proficiency model as proposed by Kilpatrick, Swafford and Findell (2001). It is suggested that this model can be incorporated into the test development structure. This article explains the five strands of the mathematics proficiency model and discusses how these strands and its respective characteristics can be used to develop a Form Two mathematics test.

The Mathematics Proficiency Model

The mathematics proficiency model is a model that describes comprehensively how students can learn mathematics successfully. The description of the model helps us to understand how students acquire mathematical proficiency and has implications on how teachers can develop that proficiency in students, how teachers can be educated to achieve that goal and how teachers can test their students' proficiency through assessments.

The term *mathematical proficiency* empowers learners with the expertise, competence, knowledge, and facility in mathematics. *Proficiency*, as defined by Kilpatrick et al., (2001) encompasses five strands namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. These strands are interwoven and interdependent which means that the strands must work together for students to learn successfully. In other words, when learning proceeds, each strand should be developed with others. Mathematical proficiency is not a one-dimensional trait, and it cannot be achieved by focusing on just one or two of these strands. It is believed that to help children acquire mathematical proficiency, these five strands need to be addressed in any instructional programme.

Conceptual understanding is the comprehension of mathematical concepts, operations and relations. It refers to the ability of students to grasp mathematical ideas, understand the importance of these ideas and see them as useful (Kilpatrick et al., 2001). For example, if students understand the idea of approximation, they are able to see the usefulness of rounding numbers in estimation. This characteristic of conceptual understanding can be used to develop a test item. An example is as follows: Estimate the value of 8216 + 699. First, the students round off 8216 to the nearest thousand which is approximately equal to 8000. Then, the students round off 699 to the nearest hundred which is approximately equal to 700. Therefore, 8216 + 699 is approximately equal to 8000 + 700 that is 8700. Another characteristic of conceptual understanding is the ability of students to relate new ideas to those ideas that they have already known as in understanding the concept of mixed numbers. Here, students relate the concept of whole number and a fraction, as the ideas that they have already known, to understand mixed number as the new idea. An example of test item to show this characteristic is as follows: Write the mixed number represented by the shaded parts in Figure 1. Students will write 3 as the whole number because there are three big squares that are shaded in full and $\frac{4}{9}$ as the fraction as there are three small squares out of 9 that are shaded in the fourth big square. Therefore, the mixed number is $3\frac{4}{9}$.

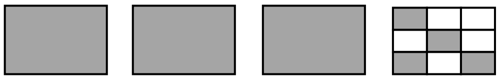


Figure 1. Representation of a mixed number.

Meanwhile, *Procedural fluency* is the skill that students should acquire in carrying out procedures flexibly, accurately, efficiently and appropriately. It refers to "knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately and efficiently" (Kilpatrick et al., 2001, p. 121). In other words, students with procedural fluency are able to carry out basic computations flexibly without always having to refer to tables or other aids. If it involves complex computations, they are able to use the calculator efficiently. Therefore, these students know the similarities and differences between methods of calculating. This characteristic of procedural fluency can be seen in the following example when these students find the sum of 398 and 235 flexibly. They are able to modify 398 as 400 less than 2 and by adding 400 and 235; they then subtract 2 from that sum. The case would be different for students without procedural fluency as they may need to use paper and pencil to solve the above addition. In addition to that, procedural fluency is especially needed to support conceptual understanding. This is illustrated when tools are used in computing, some algorithms are important as concepts in their own rights. This characteristic can be seen in the following test item: Find the value of $(-0.56)^3$ correct to 3 decimal places. Here, students need to know the concept of cube of a number as the number multiplied by itself twice before using calculators. In this case, students need to know that $(-0.56)^3$ is the same as $(-0.56) \times (-0.56) \times (-0.56)$ before computing them using calculator.

Strategic competence, on the other hand, is the ability to "formulate, represent and solve mathematical problems" (Kilpatrick et al., 2001, p. 124). This refers to the ability of generating a mathematical representation of a problem, which may be facilitated by making a drawing, or writing an equation that involves formulae. This strand is similar to what has been called problem solving and problem formulation. Students with strategic competence have broad knowledge for solving non-routine problems and not just routine problems. Routine problems are problems that students know how to solve based on their experience (Mayer & Hegarty, 1996). An example of test item that involves a routine problem showing this characteristic is finding the product of 235 and 47. This is because they know what to do and how to do it. In contrast, non-routine problems are problems for which students do not immediately know a usable solution but need to invent a way to understand and solve the problem (Kilpatrick, 2001). This characteristic of strategic competence can be used to develop test items such as in the following example of a non-routine problem: Ahmad owns a cycle shop and has a total of 72 bicycles and tricycles. There are altogether 160 wheels. How many bicycles and how many tricycles are there? An approach to this problem is by using algebra. Let b be the number of bicycles and t be the number of tricycles. Then, the students need to formulate the problem and be able to write $b + t = 72$ and $2b + 3t = 160$. The solution then yields $b = 56$ and $t = 16$.

Adaptive reasoning is the capacity for “logical thought, reflection, explanation and justification about the relationships among concepts and situations” (Kilpatrick et al., 2001, p. 129). This refers to the capacity of students to think logically that involves careful consideration of alternatives. Their ability and knowledge give reasons for their thought and justify their conclusions. Students use it to explore the many facts, procedures, concepts and solution methods and see that they all fit together in some way and make sense. In mathematics, deductive reasoning is used to settle disagreements. Agreements arise when the given answers are correct based on the series of logical steps. When facing with disagreements about a mathematical answer, students with adaptive reasoning only need to check that their reasoning is valid (Kilpatrick et al., 2001). There is no need to rely on teachers or asking opinions from friends. The characteristics of this strand can be used to develop test items. For example : Determine whether a triangle with sides 5 cm, 6 cm and 8 cm is a right-angled, obtuse-angled or acute-angled triangle. In this case, students need to find the square of the longest side, that is, $8^2 = 64$ and the sum of the squares of the other two sides, that is, $5^2 + 6^2 = 25 + 36 = 61$. Students then need to justify their conclusions that the triangle is an obtuse-angled triangle because $8^2 > 5^2 + 6^2$.

Productive disposition is the “tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off and to see oneself as an effective learner and doer of mathematics” (Kilpatrick et al., 2001, p.131). This refers to the ability of students to recognise that mathematics is sensible and useful, developing positive attitudes and gain confidence as mathematics learners. In contrast, students who have not developed a productive disposition have negative attitude towards mathematics and see themselves as incapable of learning mathematics (Riddle & Rodzwill, 2000). An example of test item to show the characteristics of this strand is as follows: How confident are you in the following situations? (i) When you measure angles using a protractor (ii) When you count 8 – 1 = ___ + 3. A. Completely confident B. Confident C. Fairly confident D. Not confident at all. Students who believe that they have the knowledge will opt for A. This type of item is normally found in student questionnaire in researches that focus on attitudes towards mathematics, beliefs about one’s own ability and beliefs about the nature of mathematics.

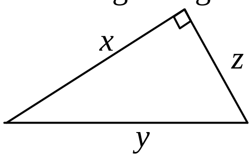
To summarise the above discussion, there are five mathematics proficiency strands as proposed by Kilpatrick et al. (2001), namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. In short, these researchers view mathematical proficiency as the ability to understand, compute, solve, and reason, and goes beyond to include disposition toward mathematics. These five strands are to work together for students to learn mathematics successfully. The knowledge of these strands together with each strands’ respective characteristics can provide guidelines to teachers in test development. This mathematics proficiency model can then be used as a foundation to develop a mathematics test.

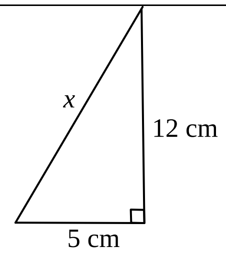
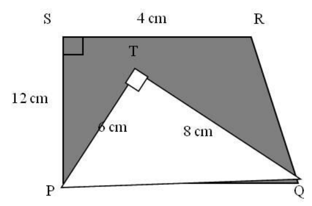
Development of Mathematics Test Items

Testing serves many important purposes. Tests are used to (1) diagnose individual student's strengths and weaknesses, (2) monitor student's progress, (3) assign grades to students (4) determine the teacher's own instructional effectiveness (5) motivate improved student, school, district, and state performance, and (6) make school and college entrance decisions (Centre for Assessment and Evaluation of Student Learning (CAESL), 2004). In accordance with learning mathematics, test can be developed to determine students' mathematical proficiency based on Kilpatrick et al. (2001) mathematics proficiency model. Table 1 shows the strands of mathematical proficiency, the characteristics of each strand and some illustrative examples of items that can be developed based on the criteria of each strand. The examples given are focused on the topic Pythagoras' Theorem of Form Two mathematics (Ministry of Education, 2004).

Table 1

Strands of Mathematical Proficiency, Characteristics of Strands, Learning Objectives, Learning Outcomes and Examples of Test Items

Strands	Characteristics of Strand	Learning Objective	Learning Outcome	Examples of Test Item
Conceptual understanding – comprehension of mathematical concepts, operations, and relations	Students are able to understand the concept of a right-angled triangle Students are able to relate new idea (Pythagoras' theorem) to the idea that they have already known (the squares of number)	6.1 Relationship between the sides of a right-angled triangle	I. Relationship between the lengths of the sides of a right-angled triangle	Write the relationship between the sides of the following triangle.  This item measures students' proficiency in conceptual understanding. This item asks students to relate the sides of a right-angled triangle. Students' ability to relate concepts is a characteristic of conceptual understanding. Students recognise the above as a right-angled triangle and relates this to the Pythagoras' theorem concept : $y^2 = x^2 + z^2$
Procedural fluency – skill in carrying out procedures flexibly,	Students are able to compute fluently numbers to find the	6.1 Relationship between the sides of a right-angled triangle	ii. Finding the length of the unknown side of a triangle	Find the value of *x* in the following right-angled triangle.

accurately, efficiently, and appropriately	squares of numbers and the square root of numbers Procedural fluency is needed to support conceptual understanding			 This item measures students' fluency in operating with squares and square roots of numbers. Students relate the sides of a right-angled triangle, and compute the value of squares of numbers. Correctly completed problem shows that students have developed procedural fluency. $x^2 = 12^2 + 5^2$ $= 144 + 25$ $x = \sqrt{169}$ $= 13$ cm
Strategic competence – ability to formulate, represent, and solve mathematical problems	Students are able to formulate problem and write equation that involves formula. In this case, the formula of trapezium and triangle. This strand is similar to what has been called problem solving and problem formulation.	6.1 Relationship between the sides of a right-angled triangle	v. Solving problems involving Pythagoras' Theorem	The diagram shows a trapezium PQRST and a right-angled triangle PTQ. Calculate the area, in cm^2, of the shaded region.  This item measures students' problem-solving abilities, which is a characteristic of students' proficiency in strategic competence. Students initially find the length of PQ using Pythagoras' Theorem.

				Then, students formulate : Area of shaded region = Area of trapezium – Area of triangle = $\frac{1}{2}(a+b)h - \frac{1}{2}bh$ = $\frac{1}{2}(4+10)12 - \frac{1}{2}(6)(8)$
Adaptive reasoning – capacity for logical thought, reflection, explanation, and justification	Students are able to explain and justify conclusions	6.2 Converse of Pythagoras' Theorem	ii. Solving problems involving converse of Pythagoras' Theorem	A scout has three bamboo stems. The lengths of the stems are 7 cm, 24 cm and 25cm respectively. The scoutmaster asks him to use the bamboo stems to form a right-angled triangle. Can he do it? This item measures students' proficiency in adaptive reasoning, in conjunction with other strands. This item asks students to reason about the properties of triangles and also assesses their conceptual understanding. Students relate the properties of a right-angled triangle. $25^2 = 24^2 + 7^2$ Then, students justify and explain that the triangle is right-angled triangle based on the above calculation.
Productive disposition – ability to see mathematics as sensible and useful, have positive attitude and confident	Students are able to perceive themselves as good at mathematics and view mathematics as useful.	-	-	How confident are you in the following situation? When you determine whether a triangle with the following sides, in cm, is a right-angled triangle. 8, 15, 17. A. Completely confident B. Confident C. Fairly confident

				D. Not confident at all This item measures the students' level of confidence in doing the above sum. This test item is especially useful when studying the relationship of students' perception in mathematics with achievement.

As a whole, the above illustrative examples show that test items can be developed based on the five strands of mathematical proficiency and their corresponding characteristics. While the fifth strand is a difficult characteristic to be measured among students, students' ability to solve such questions successfully is an indicator that they are confident. Another possible difficulty is the overlapping of one proficiency strand with another proficiency strand or strands. For example, in a test item, the proficiency strand involved could be a combination of other strands. This is expected as these five strands are not independent but are interwoven and interdependent which are complementary with each other.

Conclusion and Implication

The present conceptual paper suggests that a mathematics proficiency model can be used in test development. The use of a mathematics proficiency model in test development is seen as an advantage because test item is developed based on the model that specifically describes about proficiency in mathematics. The five strands and their respective characteristics help mathematics test developers to understand better the nature of each of the test item developed. The application of Bloom's taxonomy in mathematics test development, on the other hand, does not focus primarily on mathematics. In addition to that, the assumption that cognitive processes are ordered on a single dimension of simple-to-complex behaviour is a notable weakness in test development. This is not the case in the mathematics proficiency model that describes the strands of proficiency as interconnected and not ordered on one dimension of single-to-complex but rather is interacted with one another. This illustration is seen as more appropriate in test development. As a conclusion, it is our belief that the application of mathematics proficiency model can be used as a foundation and be incorporated in test development.

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