Development of a Module for Teaching Mathematical Problem Solving at Primary Level

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Received first draft 8 May 2019. Received reports from first reviewer (13 June 2019); second and third reviewers (23 September and 5October). Received revised draft 4 December. **Accepted** to publish 20 December 2019.

Abstract

In this paper, the researchers describe their conceptualization of module for teaching mathematical problem solving at the upper primary level on topics Measurement and Geometry. The conceptualization is based on the mathematics practical paradigm that has been used for teaching problem solving at the secondary level. One highlight of the teaching module that was developed is a set of scaffolding guide for enacting the primary mathematics problem solving lesson together with the use of the problem solving "practical worksheet" that was designed. The researchers explicate the pedagogical principles in designing the scaffolding questions in the practical worksheet. The modified practical worksheet provides teachers with a scaffold for enacting problem solving lesson. A set of four problems was chosen, the genre of which is quite uncommon for high-stake national examinations but are mathematically rich problems to be used in the upper primary mathematics curriculum. Suggestions are made on how the package can be used through the lesson plans that were developed for the lessons.

Keywords**:** Mathematical problem solving; Polya's problem solving model; Geometry; Upper primary level

Introduction

After Polya's first edition of the problem solving book "*How to Solve"* that was published in the 1940s, mathematical problem solving has received worldwide attention among the education community. Since the early 1990s, problem solving has been the focus of the Singapore mathematics curriculum for K-12, and it is still the heart of the curriculum. Despite the numerous regular curriculum revisions carried out by the Singapore Ministry of Education (MOE), problem solving remains the heart of the curriculum.

The main components of mathematical problem solving include logical reasoning, independent thinking as well as application of mathematical concepts and skills (Rahman & Ahmar, 2016). These skills and processes are the core competencies in the globalized society within the 21st Century Competencies Framework (MOE, 2015)(Figure 1). Thus, problem solving will still be relevant in mathematics education in the future.

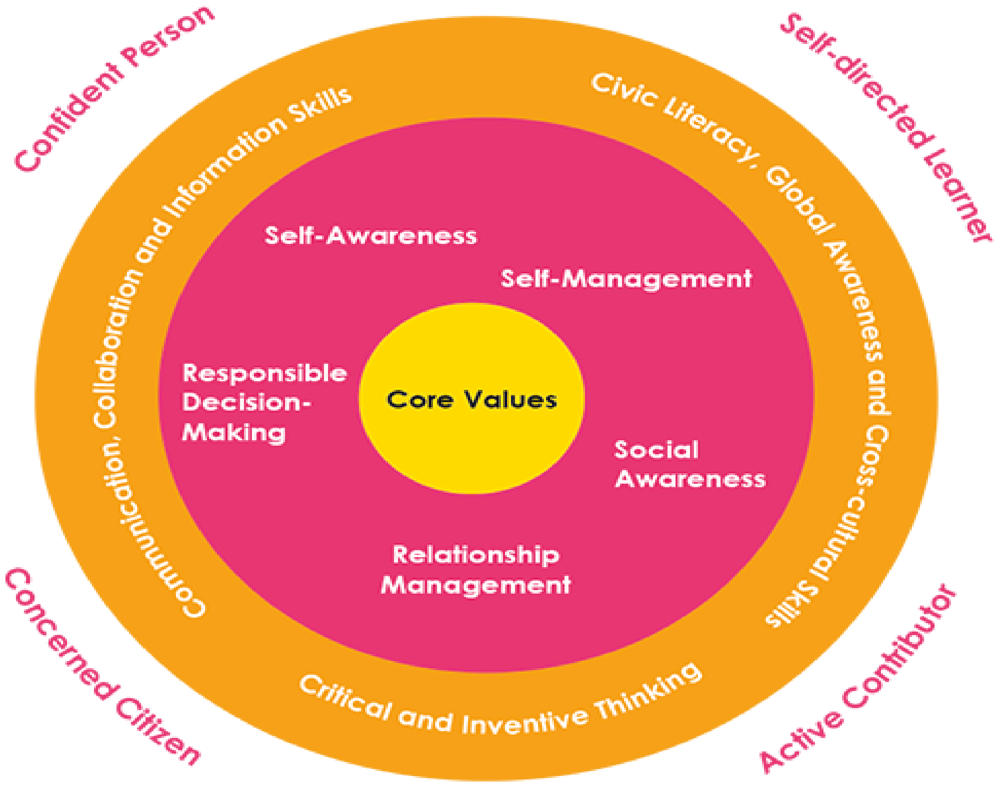


Figure 1. 21st Century Competencies Framework (MOE, 2015).

In this paper, the conceptualization of teaching problem solving and development of a problem solving teaching module tailored for upper primary students at Grade 6 are reported. The Grade 6 mathematics topics Measurement and Geometry were chosen to provide the context for problem solving. This teaching module foregrounds problem solving with the background mathematical content as its context (Lester, 1983). In other words, this module is about teaching *about* problem solving, and is distinct from most other traditional resources on teaching for problem solving, using the language of Lester (1983).

Literature Review

Background and Current State of Problem Solving

Singapore students have performed well in the various international comparative studies such as the Trends in International Mathematics and Science Study (TIMSS) as well as the Programme for International Student Assessment (PISA) of the Organization for Economic Cooperation and Development (OECD). In spite of the students' overall good performance in mathematics, there are studies which show that Singapore students generally may still not be proficient in solving unseen problems (Kaur, 2009).

In Singapore primary mathematics classrooms, anecdotal evidence shows that mathematics teachers tend to associate a strict one-one correspondence between each of the problem solving heuristics (in the curriculum document) and a mathematics problem. In addition, teachers are known to involve their students in using standard procedures to solve mathematics questions at the expense of relational understanding of the problem situation or engaging them in the full problem solving processes (Toh, Quek, Leong, Dindyal & Tay, 2011a). Moreover, due to the high-stake national examination at the end of the students' primary education, students tend to focus on the types of questions that are found in the national examinations (Toh et al., 2011a).

It is thus not surprising that the spirit of problem solving becomes latent due to teachers routinizing unseen problems into exercises, as the opportunity for students to "struggle" in problem solving is replaced by repeated practice of many similar exercises using the same problem solving heuristics. It still remains a challenge when students encounter unseen questions, as they continue to remember by rote the various approaches for specific genres of questions (Arcavi, Kessel, Meira, & Smith, 1998).

Rationale and Justification

Researchers have advocated an emphasis on the true spirit of problem solving in the mathematics classrooms, especially at the secondary level. Toh et al. (2011a) developed a module for teaching mathematical problem solving at the secondary level based on the Science practical paradigm, which they termed as "mathematics practical lessons". The mathematics practical idea was to position problem solving to the mathematics curriculum as analogous to science practical lessons to the science curriculum. Toh et al. (2011a) developed a set of scaffolding, which they called "mathematical practical worksheets" accompanying the teaching module.

Toh et al. (2011a) adopted Polya's four phase problem solving model as their theoretical framework. The authors acknowledged that in fact any problem-solving model is equally viable. However, they decided on Polya's model because it was easy to follow and it is relatively well-known. The modified version of Polya's model is shown in Figure 2. In particular, Toh et al. (2011a) renamed Polya's stage 4 (Look Back) to 'Check and Expand', in order to reflect the true spirit of Polya. Not only that, Toh et al. (2011a) explicitly highlighted the non-linear nature of the four phases by including the numerous loops within the four phases.

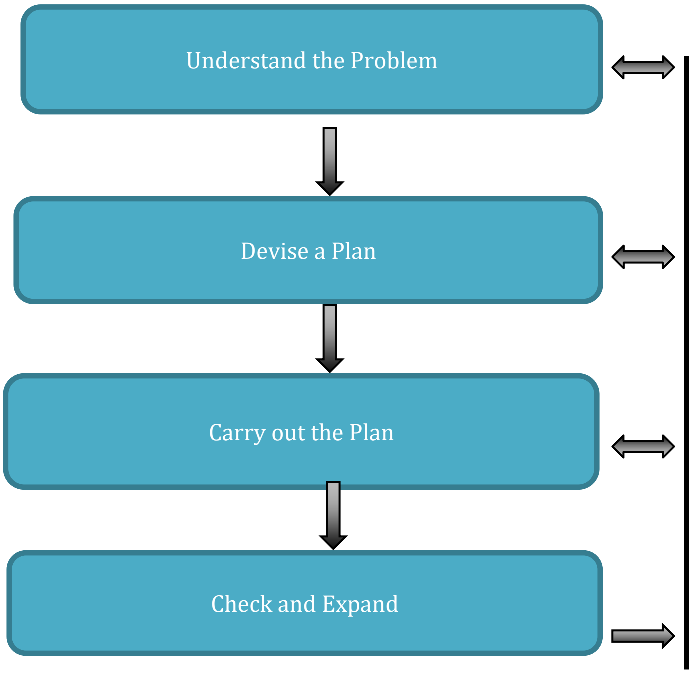


Figure 2. Polya's Problem Solving Model adapted by Toh et al. (2011a)

In translating Polya's four phase problem solving model into workable units to be used for instruction in the school mathematics classrooms, Toh et al. (2011a) included the four dimensions of Schoenfeld's framework in analyzing the complex problem-solving behavior (cognitive resources, heuristics, control and belief systems) (Schoenfeld, 1985) with Polya's problem solving model to synthesize the "mathematics practical worksheet". Appendix A shows the scaffolding questions in a condensed version of the practical worksheet.

This teaching module was conducted on one lower secondary class in each of the several Singapore mainstream secondary schools (Toh et al., 2014) and one Normal Academic students from another Singapore mainstream school (Leong, Yap, Quek, Tay & Tong, 2013). Another modified teaching module using the similar design principles has been developed and used in teaching undergraduate mathematics for pre-service secondary school mathematics teachers (Toh et al., 2013). The results from these studies generally show positive impact on student learning. In Toh et al. (2013), it was reported that the students were able to exhibit problem solving behavior in the research lessons. They were able to move to Polya's stage 4 in checking and expanding the problem. It is a common knowledge that Singapore students usually stop at giving the correct solution to a problem without moving on to Polya's stage 4. Moreover, interview with selected students shows that they appreciated the *processes* of problem solving, which was usually neglected in the usual classroom mathematics instruction.

Objectives of Study and Research Questions

Based on the above results and the positive impact such an approach has on secondary school students and pre-service teachers, It is strongly believed that a similar outcome could be achieved if a similar problem solving teaching module is developed for the primary level.

The objective of the study reported in this paper was to conceptualize and design a similar problem solving module that is workable in the primary school context. In this paper, discussion is made on the design of such a problem solving teaching module for students at the upper primary level (Grades 5 and 6) with commentaries reported based on the data collected from observation of researchers who developed this module and the responses from two primary mathematics teachers. The design process is modelled after Toh et al. (2011b, 2014) and Leong et al. (2013) in conceptualizing and designing the teaching module. It seeks to answer the following Research Questions:

(1) What are the features/attributes of a successfully implemented secondary mathematics module following Polya problem-solving methods that should be emulated at primary mathematics level?
(2) What are the aspects to be considered for the development of primary mathematics module following Polya problem-solving methods taking into account the prior knowledge and levels of achievement of primary students studying mathematics topics such as Measurement and Geometry?

Methodology and Analysis

Research Design and Development of Module

The design of the teaching module was modelled after the *Making Mathematics Practical* problem solving module described by Toh et al. (2011a). In developing the teaching module on Geometry and Measurement, each lesson was designed centering on one particular mathematics problem. Each problem chosen for the problem solving lesson illuminates one particular aspect of problem solving that is to be the focus of that lesson. The mathematical content was within the reach of the students (Grade 5 and 6 Geometry and Measurement). In selecting and adapting of the problems to be used for the lessons, two experienced primary mathematics teachers were consulted.

The design experiment approach adapted from Toh et al. (2011a) was used in this study. The researchers designed the problem solving module based on the objective of problem solving and in consultation with participating teachers. The module was then trialed in the participating schools. With the feedback obtained through lesson observations and the informal interview with teachers teaching the module, the module was refined and subsequently trialed in the participating schools. Refinement and accommodation was done after each cycle of trialing in the schools. In the teaching module described in this paper, the researchers are only at the stage of conceptualizing and designing the problem solving module for primary school mathematics lessons.

The design of the teaching module was also guided by the three principles stipulated in Toh et al. (2011a):

(1) Each selected problem should be a completely "new" problem for the students, or that the problem does not explicitly provide clue for the students to link each genre of problems with a particular heuristics (in other words, the selected problem should be of a genre that students seldom encounter);
(2) The problem should be solvable only when the solver needs to "struggle" through all the four phases of Polya's problem solving model; and
(3) The teacher's role in the lesson should be shifted away from providing students with complete solution to that of providing prompts at appropriate juncture when students are "stuck" in the problem solving process.

The problem solving teaching module consists of five lessons, each contains one practical worksheet with all the scaffolding questions (Appendix B1), a scheme-of-work (Appendix C), five lesson plans (the lesson plan of the first of the five lessons is found in Appendix D) and four selected problems with commentaries (in the subsequent section). The proposed duration of each of the five lessons is about 50 minutes.

Every lesson of the teaching module focuses on several crucial aspects of mathematical problem solving. The first lesson discusses the difference between a problem and an exercise with illustration from Geometry and Measurement. The second, third and fourth lessons focus on the different aspects of mathematical problem solving highlighted by the scaffolding questions in the mathematics practical worksheet (survey, sketch, solve and stretch). The last lesson provides a review of the entire problem solving processes through the mathematics practical worksheet. Each lesson (with the exception of the fourth lesson) focuses on one particular problem using and solving the problem using the scaffolds of the practical worksheet.

Development of Scaffolding Activities using Practical Worksheet

The scaffolding activities in the practical worksheet (which was originally designed by Toh et al. (2011a) for secondary students) were adapted for use at upper primary students at Grades 5 and 6 level. In the subsequent discussion, the practical worksheet designed by Toh et al. (2011a) will be called as "existing practical worksheet" (EPW) and the practical worksheet that was developed for the problem-solving teaching module at the primary level will be named as the "modified practical worksheet" (MPW). To begin with, it is noted that the existing practical worksheet of Toh et al. (2011a) is too lengthy and wordy for primary school students. Appendix B2 shows the modified practical worksheet that was designed based on the existing practical worksheet.

In developing the modified practical worksheet (MPW), four major modifications were made from the existing practical worksheets: (1) the use of acronym, (2) inclusion of checklists (3) use of visual representations and (4) introducing a section 'my (first) solution'.

Firstly, the four stages of Polya's problem solving model were substituted with an acronym (SSSS). The acronym was used with the intention to present the four Polya stages in a manner that is easier for primary students to remember: (1) **<u>S</u>urvey** the question, (2) **<u>S</u>ketch** your plan, (3) **<u>S</u>olve** the question and, (3) **<u>S</u>tretch** the question.

The use of acronyms is one of the many mnemonic methods which can facilitate student learning by enabling students to easily retrieve crucial knowledge (Kolencik & Hillwig, 2011, as cited in Lukie, 2015). Maccini and Ruhl (2000) (as cited in Freeman-Green, O'Brien, Wood & Hitt, 2015) also used the acronym STAR (i.e. Stop, Think, Act, Review) successfully in guiding students to learn mathematical problem solving. The proposed STAR strategy introduced a scheme for students to follow through the entire problem solving process independently. It is believed that the use of acronyms in the modified practical worksheet will be able to help students internalize and retrieve the problem solving steps easily (Miller, Strawser & Mercer, 1996) in solving problems. This acronym SSSS is specific to the four stages of Polya's problem solving model, and that it conveys to students the approach to solve problems on topics Measurement and Geometry effectively.

Secondly, in reviewing the existing practical worksheet, it was also found that the several lengthy scaffolding questions in all the four stages of the problem-solving model to be too cognitively demanding for students at the primary level. Primary school students, who are considered as 'text-participants', have not developed the fluency in reading and comprehension of such lengthy text (Winch, Ross Johnson, March, Ljungdahl, & Holliday, 2014). Winch et al. (2014) asserted that text-participants utilise images and interactive strategies to help construct meaning. Aligned with this belief, the lengthy scaffolding questions by the checklists in the EPW (in Appendix A) were replaced in the modified practical worksheet (MPW). However, most of the content within the question items used in the checklist in the MPW has been adapted from the EPW, so that the core ideas of Polya's four stages of problem solving are retained. Researchers such as Kingsdorf and Krawec (2016) have affirmed the importance of checklists as they allow students to monitor their problem solving learning independently and regularly.

Thirdly, phase two of Polya's problem solving model (devise a plan) was modified to explicitly getting the students to *sketch the question* instead. The use of visual representations, especially for the topics Measurement and Geometry, can facilitate student learning, as they are likely to achieve a better understanding by associating visual representations with mathematical ideas (Furner, Yahya & Duffy, 2005). Since the researchers focused on Measurement and Geometry as the mathematics topics with context on engaging students in problem solving, it is believed that to interpret explicitly phase two as "sketching the question" is crucial. This interpretation will likely facilitate students to visualize problems through pictorial representations. Drawing deepens students' understanding of mathematics problems, especially for Measurement and Geometry. It will also likely to be leading them to build their competence in explaining and understanding mathematical concepts, thereby building their confidence in problem solving.

Lastly, a section entitled "My (first) solution" is included under the section 'Solve it' in the modified practical worksheet. This is similar to Leong et al.'s (2013) adaptation of the EPW to teach problem solving to lower secondary Normal Academic students in one Singapore mainstream school. The objective of this inclusion was to lead students to appreciate that the solution written in the initial stage need not be (and usually is not) the final solution (Leong et al., 2013). This resonates with Toh et al. (2011a) that problem solving is neither a linear nor sequential process; students need to build up the habit of monitoring and assessing their actions progressively when solving a problem (Phillips, Clemmer, McCallum & Zachariah, 2017). In Stage three of the MPW, the researchers reinforce in students the importance to review and revise their solution during problem solving.

Discussion of Findings on Implementation Procedures with Exemplars and Commentaries

This section discusses the analysis of data in response to Research Question (RQ) 1 and 2 as aforementioned.

Problem Selection and Criteria for Problem Construction

In response to RQ1, 'What are the features/attributes of a successfully implemented secondary mathematics module following Polya problem-solving methods that should be emulated at primary mathematics level?', elaboration will be made on how problem was selected and what are the criteria for problem construction in the module that was developed to teach Mathematical problem solving at primary level,

In developing this teaching module, the following three criteria adapted from secondary mathematics module were again used to construct the problems to be used in the module. The problems that are used for the module are:

(1) Not commonly seen in the usual instructional resource or national examination papers;

(2) Those for which the solutions of which must not be easily obtained, but still within students' cognitive "resource" (Schoenfeld, 1985); and

(3) Problems that demand the solvers to apply their reasoning skills and mathematical content knowledge in order to solve them (Aydogdu & Kesan, 2014).

However, the significant difference between secondary and primary mathematical problem-solving as summarized in the following Table 1 is also elaborated.

Table 1

Differences between Secondary and Primary Mathematic Module to Teach Problem-solving

Problem-Solving Processes and Scaffolding	**Secondary Module (Toh et al., 2011a)**	**Primary Module**
Polya's Stage One: Understanding the problem	Using "heuristics" to understand the words, and emphasis on individual effort to understand the problem.	Emphasis on reading the questions carefully, highlighting key words, and clarification with teachers and classmates.
Polya's Stage Two: Devising a Plan	The full list of heuristics that is proposed in the syllabus document.	Emphasis on six heuristics.
Polya's Stage Three: Carry Out the Plan	Emphasis on students solving the problems and voicing out their "control", and that it may take	Similar emphasis that it may need more than one attempt to solve a problem correctly.

	more than one attempt to solve the problem correctly.	However, students are not expected to voice out their "control" in solving the problem
Polya's Stage Four: Check and Expand the problem	Emphasis on both checking the reasonableness of solution, and of expanding the problem.	Only emphasize on checking the reasonableness of the solution.
Language used in the lesson	Use the vocabulary of the problem solving literature.	Simplify the language, e.g. Use SSSS as an acronym for the four stages of Polya's model.
Scaffolding	EPW uses facilitating sub-questions as scaffold.	MPW uses checklists instead of the sub-questions.

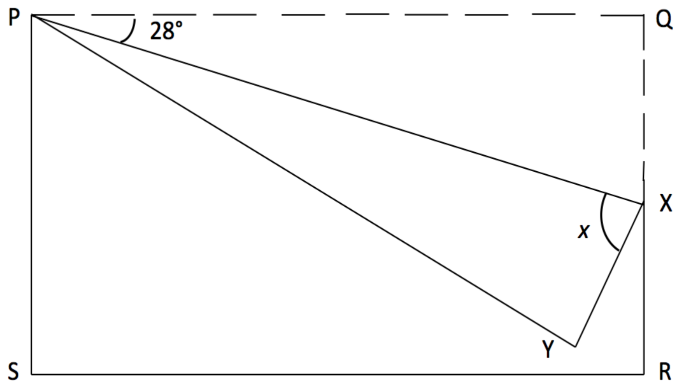
Aspects to be Considered for the Choice of the Problems

This section illustrates four exemplars in response to RQ2, 'What are the aspects to be considered for the development of primary mathematics module following Polya problem-solving methods taking into account the prior knowledge and levels of achievement of primary students studying mathematics topics such as Measurement and Geometry?'

Presentation is made on the problems that were eventually used for the teaching module with commentaries compiled from respondents of this study (i.e. observation of the researchers who developed this module in consultation with two experienced primary mathematics teachers). The Scheme-of-Work (Appendix C) and Lesson Plans (Appendix D) are also elaborated with exemplars appended.

Mathematical Problem-Solving Exemplar 1

The following figure shows a rectangular piece of paper PQRS folded along PX. It is known that $\angle QPX = 28°$. *Find x.*



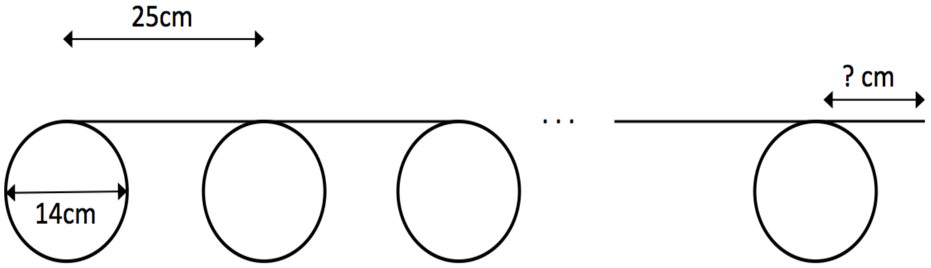
Commentary:

Exemplar 1 is used in the teaching module to illustrate what distinguishes a mathematics problem from an exercise (it is generally accepted among the mathematics education community that an "exercise" is a task which is routine, that is, its solution is easily forthcoming based on what the students have learnt from the usual classroom instruction). In solving this problem, students need to use the property of the preservation of angles. Note that at the primary level, students are not required to know congruency and similarity, an advanced geometry concept covered only at the secondary level.

The Piagetian cognitive development theory suggests that most students at the age of upper primary level (age 10 to 12) still function at the concrete operational stage. They have not fully developed logical thinking ability and are likely to require assistance (e.g. by using concrete material to act it out) to discover the property of angle preservation required in this problem. Thus, teachers' appropriate use of scaffolding in the modified practical worksheet will be useful to facilitate them to solve this problem systematically. By using this problem, teachers can bring students to realize that there are mathematics problems for which the solution might not be obtained directly. Thus, the use of Polya's problem solving model, as facilitated by the scaffolding in the modified practical worksheet, will be useful for problem solving.

Mathematical Problem-Solving Exemplar 2

John made identical circles by bending a wire as shown below. The diameter of each circle is 14cm. The length of the wire is 11m. The distance between two consecutive centers is 25cm. What is the length of wire left after forming the last circle? (Take $\pi = \frac{22}{7}$ *if necessary)*



Commentary:

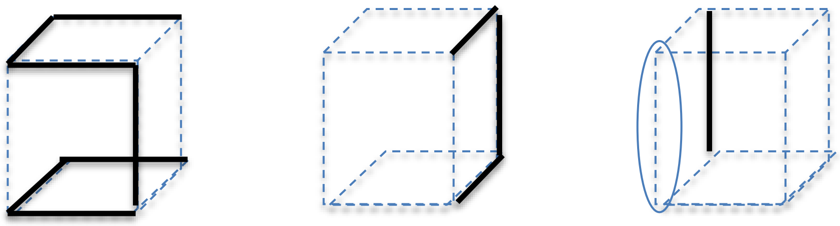
Problem 2 highlights to students the importance of fully understanding a mathematics task before attempting to solve the task. Students may be overwhelmed by the several pieces of information, mathematical terminologies and values that are presented in the problem. The researchers used this problem in the module to highlight to students the importance of understanding *all* information provided by the problem before even attempting to solve it.

Mathematical Problem-Solving Exemplar 3

There are seven pieces of wires with lengths 7m, 6m, 5m, 4m, 3m, 2m and 1m. What is the smallest number of pieces of wires used to make a 1m by 1m by 1m wire cube without any overlapping sides?

Commentary:

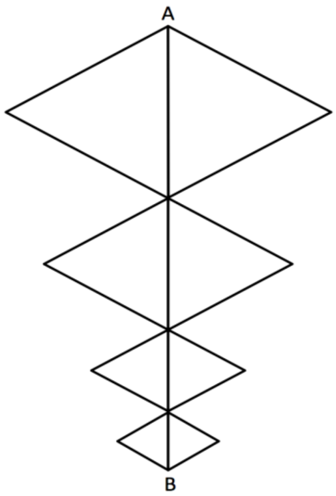
Problem 3 highlights the importance to sketch the problem in order to solve it. The problem does not provide students with much information, hence it needs the solvers to plan and use the trial-and-error heuristics in order to solve the problem. As an illustration, a student may attempt to use the 7m, 3m and one 1m wires to form parts of the cube. In this case, it is not possible to use any remaining wires to complete the sides of the cube without overlap (see figure below). Thus, the following way to form the cube is incorrect.



It builds students' logical thinking and spatial visualization, as they are trained to make sense of how the cube is formed with the given wires with specific lengths after they have fully understood the problem. Even after the students have obtained their answers, this problem forces them to check whether their answer is the minimum by checking other possible cases.

Mathematical Problem-Solving Exemplar 4

Mary had 2m of wire. He used some of the wire to bend into the shape as shown below. He formed 8 equilateral triangles and the length of AB is 37cm. How much of the wire was left?



Commentary:

Problem 4 was selected to enable students to experience all the four stages in Polya's problem solving model as scaffolded by the modified practical worksheet. It consists of several mathematical terminologies and quantities for students to make sense before they can begin solving the problem. In the proposal, teachers were encouraged to lead students to solve this problem by going through the entire process of problem solving, although the researchers are cognizant that the same problem may also be solved directly by using algebra.

Scheme-of-Work and Lesson Plans

Appendix C is appended with the proposed scheme-of-work, which provides an overview of the flow of the five lessons of the problem solving teaching module. The researchers present the lesson focus, specific learning objectives and the suggested tasks and activities that teachers can use in their lessons.

The first lesson emphasizes on understanding the difference between a problem and an exercise, using exemplar 1 as an illustration. It is believed that this lesson is crucial, as students need to recognize mathematics problems as situations in which they need to visit the entire problem solving processes that was presented in the practical worksheet. Note that the MPW that was developed for this teaching module is meant to be eventually internalized by students, so that they will be able to handle a (non-routine) problem like a mathematician when they encounter one. The MPW is not meant to be a series of tedious rituals to follow when solving questions for which the solutions are immediately forthcoming. The next three lessons deal in greater depth all the four Polya stages.

The second lesson discusses the first Polya stage (that of Surveying the question) by means of exemplar 2. The third lesson presents the second and third Polya stages (that of Sketching and Solving the question) using the context of exemplar 3. The fourth lesson highlights the fourth Polya stage (of Stretching the question), building on discussion the first three exemplars. The last lesson provides an opportunity for students to review all the four Polya stages by attempting to solve exemplar 4.

Appendix D is also appended with a sample lesson plan of the first of the five lessons in the teaching package, which is proposed to last 50 minutes. It provides suggestion on how the suggestion should be enacted with specific details. The problems selected for each lesson have been chosen to meet the learning objectives of that lesson as indicated in the scheme-of-work in Appendix C. Teachers conducting the lesson are encouraged to adhere to the time frame and the problems for each lesson.

A summary of the lesson and exemplars used, objectives, the problem solving processes and the "cognitive resources" (Schoenfeld, 1985) is shown in Table 2.

Table 2
Exemplars versus the Lessons, the Polya Process, Problem Solving Heuristics and Cognitive Resources of the Teaching Module

Ex. No.	Lesson	Polya's stage	Problem solving heuristics	Cognitive resource
1	One & Four	Distinguish between a problem and an exercise.	Drawing a diagram; Act it out (to discover angle preservation)	Angle sum of a triangle
2	Two & Four	Stage 1: Understand a problem	Act it out (to discover the repetition unit); Simplify the problem	Formula for circumference of a circle; Multiplication as repeated addition.
3	Three & Four	Stage 1: Understand a problem Stage 4: Check their answer	Trial-and-error (for the choice and orientation of the wires); Act it out (to discover which choices and orientations are possible)	Terminologies involving a cube: vertices, edges and sides.
4	Five	All four stages. In particular Stage 2: Devise a plan Stage 3: Carry out the plan	Trial-and-error; Act it out (recognize that the total perimeter of the compound is three times the length of *AB*); Simplify the problem	Perimeter of a triangle; Solving simple equation (using bar model method).

Conclusion

This paper presents the development of a mathematical problem solving teaching module for students at the upper primary level, focusing on primary mathematics topics Measurement and Geometry. This module was conceptualized and designed based on a problem solving teaching module for secondary school students designed by Toh et al. (2011a). The module emphasizes the use of scaffolding through a modified practical worksheet. The intent and the underlying intent of this problem solving teaching module has been described in this paper. However, to determine the efficacy of this teaching module, the enactment of the module needs to be carried out in an authentic mathematics classroom. The researchers also note that the assessment strategy accompanying this teaching module needs further work. The fundamental idea of our proposed assessment strategy is that, in addition of assessing the students' correctness of the solution, their processes of problem solving must also be assessed. As it is well known, assessment drives the way students learn a subject. Adapting the assessment strategy in Toh et al. (2011b) for the current module is still part of work-in-progress at the current stage.

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Appendix A:

The scaffolding questions in the condensed practical worksheet (Toh et al., 2011)

Problem

Print your problem here

Instructions

1. You may proceed to complete the worksheet doing stages I – IV.
2. If you wish, you have 15 minutes to solve the problem without explicitly using Polya's model. Do your work in the space for Stage III.
   - If you are stuck after 15 minutes, use Polya's model and complete all the stages I – IV.
   - If you can solve the problem, you must proceed to do stage IV – Check and Expand.

I Understand the problem

(You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.)

(a) Write down your feelings about the problem. Does it bore you? scare you? challenge you?

(b) Write down the parts you do not understand now or that you misunderstood in your previous attempt.

(c) Write down your attempt to understand the problem; and state the heuristics you used.

Attempt 1

II Devise a plan

(You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.)

(a) Write down the key concepts that might be involved in solving the problem.

(b) Do you think you have the required resources to implement the plan?

(c) Write out each plan concisely and clearly.

Plan 1

III Carry out the plan

(You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.)

(i) Write down in the *Control* column, the key points where you make a decision or observation, for e.g., go back to check, try something else, look for resources, or totally abandon the plan.

(ii) Write out each implementation in detail under the *Detailed Mathematical Steps* column.

Detailed Mathematical Steps	*Control*
<u>Attempt 1</u>	

IV Check and Expand

(a) Write down how you checked your solution.

(b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of.

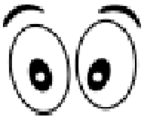
(c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

Appendix B1:
Modified Practical Worksheet (MPW) (condensed form)

Modified Practical Worksheet (condensed form)

Problem

Print the problem here.



Survey the question

☐ I read the question at least once.

☐ I highlighted important words in the question and made some notes about the question.
I am unsure about ____.

☐ I clarified with my classmate(s) and/or teacher about the part I do not understand.

☐ I understand what the question wants me to find.

Page Two of Practical Worksheet

Sketch your plan

Plan your work here! You may draw to help you plan your work.

Tick the possible heuristic(s) you may use: (you may choose more than 1)

( ) Drawing a diagram
( ) Trial and error/ Guess and check
( ) Acting it out
( ) Working backwards
( ) Simplifying the probem
( ) Considering special cases

Page Three of the Practical Worksheet is a blank page for students to sketch their plan.

Solve the question

Write your first solution here! If you think you need to revise your plan, you may refer back to 'Setch your plan' before returning here again.

My (first) try!

Page Four of Practical Worksheet

Stretch the question

☐ My answer is reasonable and makes sense in the context of the question.

Appendix B2:
Modified Practical Worksheet (MPW)(Concise version)

Problem

Print the problem here.

Survey the question

☐ I read the question at least once.

☐ I highlighted important words in the question and made some notes about the question.
I am unsure about ____.

☐ I clarified with my classmate(s) and/or teacher about the part I do not understand.

☐ I understand what the question wants me to find.

Sketch your plan

Plan your work here! You may draw to help you plan your work.

Tick the possible heuristic(s) you may use: (you may choose more than 1)

( ) Drawing a diagram
( ) Trial and error/ Guess and check
( ) Acting it out
( ) Working backwards
( ) Simplifying the probem
( ) Considering special cases

Solve the question

Write your first solution here! If you think you need to revise your plan, you may refer back to 'Setch your plan' before returning here again.

My (first) try!

Stretch the question

☐ My answer is reasonable and makes sense in the context of the question.

Appendix C: Scheme-of-work

Lesson	Lesson Focus	Specific Learning Objectives	Suggested Task & Activities
1	Exercise or problem?	Students should be able to: - differentiate between an exercise and a problem in mathematics - state the key features in each phase of problem solving	- Students will be given two questions and have to note the differences between the two questions (one being a routine problem and the other a non-routine problem) - Teacher distinguishes between the routine problem and non-routine problem - Teacher models the solution to the non-routine problem - Teacher will introduce Polya's model of problem solving through the practical worksheet and guide students on how to complete it for the non-routine problem.
2	Survey the question	Students should be able to: - identify key sentences in a problem - understand the purpose of the problem and what is required from the problem	- Students will work on a problem that emphasizes on section 1 of the practical worksheet. - Teacher will highlight the importance of understanding the problem first before attempting it.
3	Sketch and solve the question	Students should be able to: - use pictorial representations to assist them in solving the problem - state and understand what heuristics are used for - utilize the practical worksheet to solve the problem	- Teacher will highlight the uses of sections 2 and 3 of the practical worksheet where students have to sketch their plan and solve the problem. - Teacher will mention explicitly what heuristics are and provide examples of heuristics in the syllabus that are relevant. - Student will work on a problem that emphasizes on sections 2 and 3 of the practical worksheet.
4	Stretch the question	Students should be able to: - check the accuracy of their solution	- Teacher will emphasize the purpose of checking for their answers for accuracy and how to do it apart from checking for calculation error. - Students will attempt to check the accuracy of their answers to previous problems.
5	The Mathematics Practical	Students should be able to: - appreciate the use of the mathematics practical to help them in the problem solving process - utilize the practical worksheet to solve a mathematical problem	- Students will complete a problem using the practical worksheet independently.

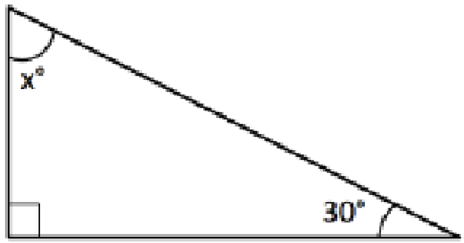
Appendix D: Detailed Lesson Plan

This section provides the detailed lesson plans of the first mathematics practical lessons.

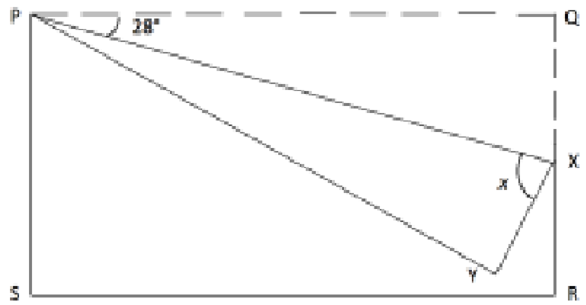
Lesson 1: Exercise or problem?

Exercise and problem: Finding unknown angles (15 mins)
Teacher shows the two questions and informs students they will have 15 minutes to complete the two questions in pairs.

(1) Find x.



(2)
The figure shows a rectangular piece of paper PQRS folded along PX. ∠QPX = 28°. Find x.



Discussion: (Total: 25 mins)
<u>Differentiating between a problem and an exercise</u>
Teacher asks students about their thoughts to the two questions and asks them to identify any differences in their workings between the two questions.
Note: Question (1) is an exercise, as students should have seen similar questions and the solution can be obtained easily and directly. Question (2) is a problem as it is a non-routine question and the solution is indirect and requires logical thinking.
Teacher explains the difference between a problem and an exercise for students.

<u>Problem solving</u>
Teacher gets student responses for question (1) and clarifies that it is as an exercise.

Teacher gets students to share their answers for question (2) and invites some students to present their working on the board. Students may explain their solution to the class.

Next, teacher models the solution of question (2) by going through the first 3 sections in the practical worksheet (survey, sketch, solve) without mentioning explicitly the steps. For this problem, teacher may use a rectangular piece of paper to illustrate the problem.

Teacher emphasizes verbally the checklists used for each section of the practical worksheet (especially for section 1: survey the question) and highlights the importance of the individual steps in solving the problem.

Teacher uses her answer and student's answer on the board to check if the answers are the same and if it is reasonable in the context of the question. This covers section 4 in the practical worksheet (stretch).

Closure:
(10 mins)
Teacher reviews the characteristics of a problem and mentions that it may be tedious and challenging to approach a problem without having a plan. Students will be introduced to a worksheet that can help them to approach problem solving in a systematic manner.

Teacher shows students briefly the practical worksheet and relates her workings modeled on the board to each phase in the practical worksheet.