Enacting a Problem-Solving Lesson using Scaffolding to Emphasize Extending a Problem

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Received first draft 16 July 2021. Received reports from first reviewer (28 July) and second reviewer (6 August); Received revised draft 22 November. **Accepted** to publish 25 November 2021.

Abstract

In this paper, we describe our conceptualization of teaching mathematical problem-solving at the upper primary level, emphasizing Polya's Stage Four in extending a problem. Geometry is used as a context of the presentation. The objective is to engage pupils more metacognitively in their problem-solving process. By reviewing existing education literature, features that will support authentic problem solving were identified. The frameworks explored in this study include Polya's 4-step problem-solving model, Schoenfeld's framework, and the synthesis of the two frameworks through "Making Mathematics Practical" which utilize an extensive use of teacher scaffolding. The proposed scaffolding stresses pupils to problem solve beyond finding a solution as well as independently check and expand the given mathematics problem.

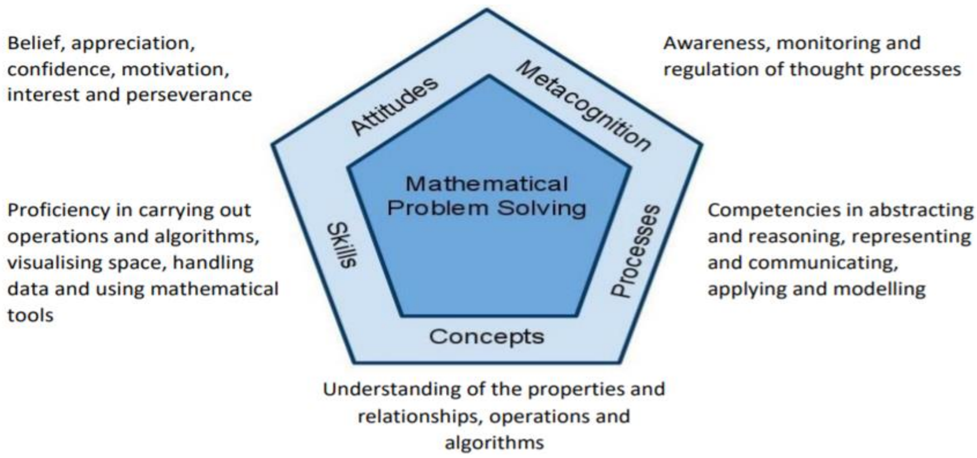
Keywords: *Singapore mathematical problem solving; Polya's problem-solving model; Scaffolding; Geometry; Upper primary level*

Introduction

Background and Overview

Since the early 1990s, problem-solving has been the heart of the primary mathematics curriculum. As with all other mathematics curricular at the secondary and pre-university levels, the primary mathematics curriculum in Singapore is guided by the Mathematics Framework (Figure 1), the guiding framework for all mathematics subjects from K-12.

Figure 1 Ministry of Education (MOE) Mathematics Framework (MOE, 2020)



According to the mathematical problem-solving framework, every student should engage in the five-inter-related components of concepts, skills, processes, metacognition, and attitudes to become a good problem solver. A good problem solver is competent in devising strategies to solve problems, logically justifying claims, writing, as well as critiquing mathematical explanations and arguments in clear communication of thoughts (MOE, 2020). These characteristics of a good problem solver are aligned to the 21st Century Competencies where the student is engaged in the skills of inventive thinking, reasoning, critical thinking and communication (MOE, 2021) (Figure 2).

Figure 2 21st Century Competencies Framework (MOE, 2021)

Problem Statement

In this paper, the researchers operationalize a *problem* as a mathematical task in which the solver does not have a readily accessible procedure that guarantees a solution, but the solver is motivated to solve the task (Lester, 1983).

Researchers' analyses of the performance of the Singapore primary school pupil in the Trends in International Mathematics and Science Study (TIMSS) revealed that their problem-solving abilities are inconsistent across mathematical domains (Dindyal, 2006). The pupils are generally not able to translate their problem-solving abilities with unfamiliar problems (Beckmann, 2004).

Furthermore, how problem-solving can be enacted in the classroom remains implicit in the curricula documents. As such, the approaches to problem-solving instruction are dependent on interpretations by individual schools and thus are not standardised across the mathematics classrooms (Lee et al., 2014). With the aim of guiding problem solving instruction, Chapters 4, 6, and 12 of the 2009 Yearbook of the Association of Mathematics Educators (Kaur et al., 2009) suggested note-taking, problem posing, and open-ended tasks respectively as problem-solving strategies a teacher can employ. It is apparent from Table 1, a listing of the problem-solving affordances of each pedagogy are not identical. There is an implication that the choice of pedagogy may thus result in pupils not performing problem solving as desired.

Table 1 Analysis of Problem Solving Pedagogies

Problem Solving Pedagogy	Affordance
Note-taking (Albert et al., 2009)	• Provide directions that aid pupils seek relevant information to use when problem-solving • Metacognitive awareness of their thought process
Problem-Posing (Yeap, 2009)	• Developing a concept • Provide direction by identifying intermediate questions within a problem
Open-ended tasks (Yeo, 2009)	• Challenge the belief of only one right prescribed solution. • Alternative solution opportunities for quick checks of pupils' thinking and conceptual understanding

At the primary level, teachers conducting problem-solving lessons generally interpret problem-solving as the exclusive teaching of heuristics and the routinizing of problems into exercises (Toh et al., 2011a). The teachers' effort of routinizing problems and the almost exclusive teaching of heuristics in the mathematics classroom contradicts the overarching goal of the Singapore mathematics curriculum to work in the context of "non-routine, open-ended and real-world problems" (MOE, 2006, p.6). Yeo (2018) recognized that the questions in Singapore primary school textbooks, worksheets, and assessments were mostly closed-ended questions and not mathematical problems where the purpose of the worked examples in textbooks are to familiarise the pupils with the procedural skills that have just been taught in the lesson so that they can practise the procedures in the subsequent textbook exercises.

In addition, studies inform that students are metacognitively unaware of their problem-solving processes due to the drilling practice in schools (e.g., Ng & Lee, 2005). Few pupils can display an awareness of which problem-solving stage they were at and the purpose of their steps beyond the generic reason of problem-solving (Wong, 2007). When they can solve the problems, the students attributed their performance to their familiarity with the questions. This suggests that they problem-solve based on prior routines and that they are not metacognitively sensitive to their thought process when they solve problems. This is partly due to the textbooks used as they do not explicitly inform the pupils which phase of problem-solving they are at but over-emphasize on the heuristics despite being the primary teaching resource in Singapore schools (Ng, 2004).

Problem-solving Lesson Package: Rationale with Evidence

Authentic problem solving happens when the individual engages with authentic tasks that model real-world practice (Cho et al., 2015). Reflecting the real-world context, authentic mathematical problems are non-routine problems with multiple plausible solutions (Toh et al., 2019). Authentic problems focused on content-specific features effectively promote particular concept development and elicit higher-order thinking (Cheng & Toh, 2015). However, anecdotal evidence in the Singapore mathematics classrooms shows that pupils are not engaged in solving authentic problems despite problem-solving being the focus of mathematics education. The routinization of problem-solving has impeded the practice of authentic problem solving and the formation of meaningful relations, which is evident in them not performing in unfamiliar problems.

There is a lack of acceptance and implementation of mathematical problem solving by pupils and teachers, as observed by the call of Toh et al. (2008, 2011a) for a shift in the teaching of problem-solving. This paper seeks to clarify how problem-solving can be enacted in the classroom with an exemplary package. This package is conceptualized after an extensive literature review on local problem-solving teaching practices was carried out. It is inspired by present-day problem-solving packages for secondary school pupils in Singapore to bridge the lack of standardization across the nation.

The development of pupils' problem-solving ability is further hindered by high-stake national examinations, resulting in pupils focusing on examinable content only (Toh et al., 2011a). So, problem solving experiences might not have been emphasized in the mathematics lessons, and students have been trained in applying problem-solving heuristics within the limited scope of routine examination questions.

Researchers have cautioned that heuristics alone are insufficient to develop problem-solving ability in pupils. According to Schoenfeld (1985), cognitive resources, metacognitive ability of control, and beliefs are the other components equally important with heuristics for successful problem solving. A metacognitive-based problem solving instruction was trialled on Primary Four pupils and it was revealed that the pupils were better supported in solving non-routine mathematics problems when they were metacognitively aware of their problem solving process. Lee et al. (2014) attributed the lack of metacognitive awareness among students to inappropriate application of heuristics and a failure to completely understand the problem context, which resulted in little success in complete problem problem-solving. This indicates a need to shift away from present-day routinized teaching of heuristics towards heightened metacognition of pupils in the learning and application of mathematical content, skills, and the process of problem-solving.

Toh et al. (2011a) considered successful problem solving the complex interaction of these four components. This reflects the Mathematical framework whereby pupils engage in the five inter-related components of *Attitudes* representing belief systems, *Metacognition* representing control, *Skills*, and *Concepts* for cognitive resources and *Processes* overlapping with heuristics.

Research Questions

Considering that problem-solving experience starts from the primary school mathematics classroom, it is necessary to extend the research to the primary school level. Hence, this research aims to conceptualize a problem-solving lesson package that teaches primary school students about problem-solving in its true spirit. The design of the package seeks to answer the following questions:

1) What are the features of an authentic problem-solving package that should be included in this package to focus students' problem-solving process on the 'Checking and Expanding' phase of their solutions?
2) How can an authentic problem-solving lesson focusing on Check and Expand be enacted in the primary school classroom?

Literature Review

Polya's Problem-solving Model

Polya's four-phase problem-solving model is well-known. His original conception of the fourth stage of his problem-solving model, which he named 'Look Back', encompassed the consideration of alternative solutions and representations, re-examining the solutions for a more efficient strategy and the extension of the solution to other related problems. Toh et al. (2011a) renamed this stage as 'Check and Expand' to reflect Polya's explicitly. At this stage, pupils are expected to take ownership and reflect for further mathematical content learning (Toh et al., 2019). However, most of the studies have focused on learning to problem solve (Polya's stages One to Three) and little about learning through problem-solving (Stage Four). The primary interpretation of 'Check and Expand' is skewed towards checking the correctness or reasonableness of the answer instead of the original conception. This is also representative of the use of Stage Four by pupils, despite existing literature (e.g., Anghileri, 2006) showing that Check and Expand can help pupils get better at problem-solving.

'Check and Expand' is most neglected and almost never enacted in mathematics classrooms (Wong, 2007), especially in the primary school mathematics classroom. Mostly, the enactment of 'Check and Expand' stops only at pupils checking for careless mistakes. Pupils are rarely engaged in exploring alternative solutions or generalizations to further their learning (Dindyal et al., 2012). According to Chan (2011), Singapore pupils generally believe that only one "correct" solution exists for every problem. This belief may have caused their prematured ending of their problem-solving process after a correct solution has been obtained. This belief was also found among teachers; as reflected by a teacher participant from an observation made of teachers conducting this stage (Leong et al., 2011). The teacher defended his position of not focusing on this stage as much as the other stages as he believed that there was only one solution to a particular problem. The extent of buy-in from the teacher participants in the study influenced the emphasis on this stage and whether they approach 'Check and Expand' according to Polya's original intention. It postulates that the belief of there only being one correct solution is a learned belief propagated by the teachers.

Enactment of problem solving lessons in adapting Polya's problem-solving model

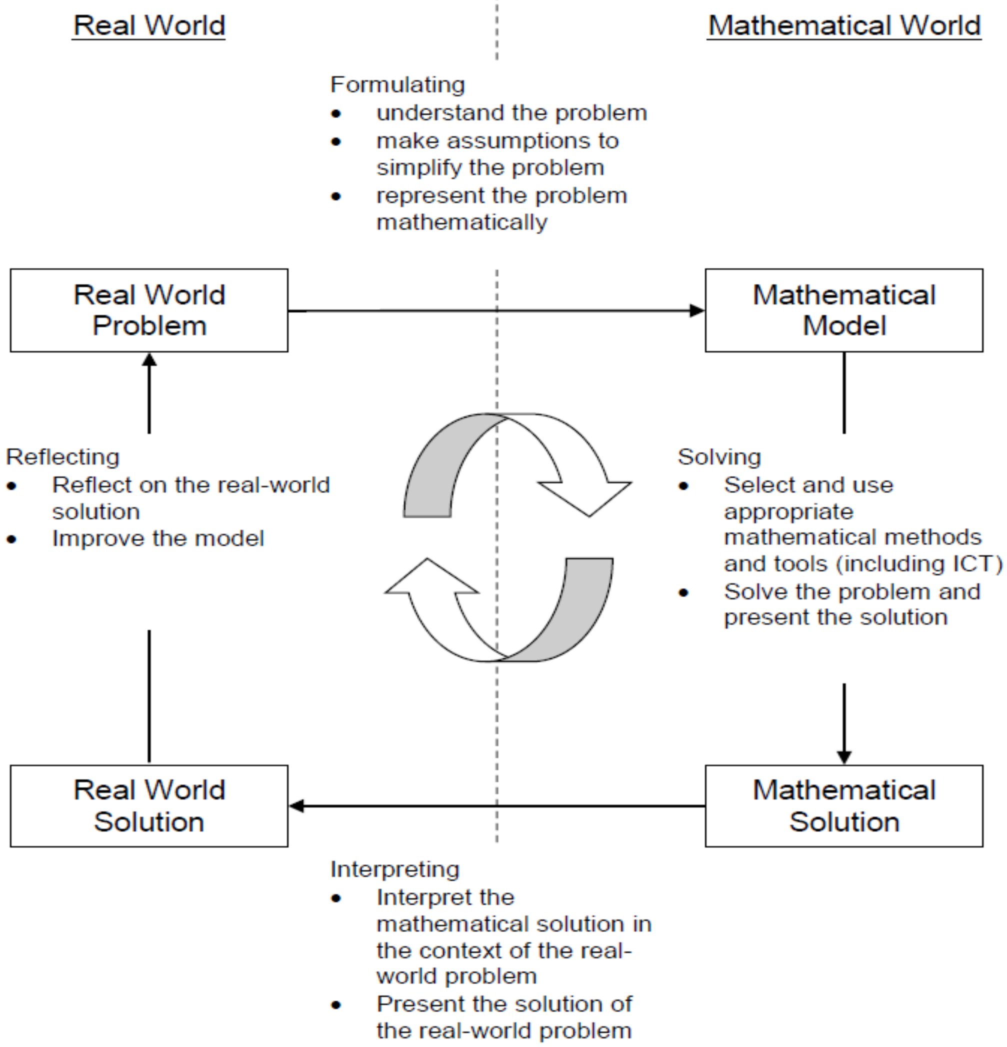
Toh et al. (2008, 2011a) adapted Polya's four-phase problem-solving model and Schoenfeld's Framework in developing a problem-solving module, using the Science Practical Paradigm. In the module, scaffolding was an important component for the enactment of the problem-solving lessons. The scaffolding manifested in a set of scaffolding the authors termed as "mathematics practical worksheets".

The teaching module designed by Toh et al. (2011a) was conducted on lower secondary classes in several Singapore mainstream secondary schools (Dindyal et al., 2012), modified for one Normal Academic class from another mainstream secondary school (Leong et al., 2013), and undergraduate pre-service secondary school mathematics teachers (Toh et al., 2013). In these studies, after learning the problem-solving approach, most of the participating students could complete the problem-solving processes and demonstrate the fourth stage of Check and Expand. The middle ability secondary school students who form the majority student profile appreciated the full problem-solving process, especially the fourth stage they originally perceived as beyond their ability. This buy-in was mirrored by the lower ability pupils who were originally resistant to learning the entire problem solving process (Toh et al., 2011b).

The approach by Toh et al. (2011a) has been further adapted in developing problem-solving packages on the topic Whole Numbers (Liang & Toh, 2018) as well as the topic Measurement and Geometry (Yong & Toh, 2019) at the primary level. The modifications tend to focus on simplifying the scaffolding worksheets by paraphrasing the labels for simpler understanding and highlighting the use of heuristics in Polya's Stages One to Three. In the two modules for the primary school pupils, there was a clear goal of teaching pupils how to solve problem up to Stage Three of Polya's problem-solving model. They did not elaborate on Stage Four, which was interpreted by both papers as checking for accuracy of their answers to the problems.

Another problem-solving package introduced in Singapore schools is mathematical modeling. MOE (2012) introduces mathematical modeling within the *processes* of the Mathematical Framework as an alternative to Polya's Model, where it is the "process of formulating and improving a mathematical model to represent and solve real-world problems." When students solve problems through mathematical modeling, they start with a real-world problem as well as using mathematical representations and tools to solve the problem (Chan et al., 2019). Figure 3 summarises the mathematical modeling process, whereby the modeler refines a mathematical model for a given situation in multiple cycles of construction, evaluation, and revision (Chan, 2009).

Figure 3 Mathematical Modelling Process (MOE, 2012)



A study by Chan (2009) found that the Primary 6 pupils (of age 12) were capable of doing mathematical modeling, including the metacognitive processes of inquiring and selfmonitoring. However, they similarly demonstrated the lack of Check and Expanded. They stopped their modeling once they had obtained a plausible solution. Deliberate teacher scaffolding was needed to provoke the pupils into reanalysing their models and adjusting their models based on modified conditions, which is the checking and expanding of their solution according to Polya's model. The pupils' mathematical modelling would have stopped before the phases of evaluation and revision of their model without teacher scaffolding points to the importance of teacher facilitation for the pupils' mathematical modelling. This finding is also supported by Chan et al. (2019), whose study shows that developing the teachers' capacity to design and facilitate mathematical modelling should be the main priority for the actualisation of the mathematical modelling curriculum in the primary classroom.

Mathematical modelling remains to be rare in the primary mathematics classroom (Chan et al., 2018). Although mathematical modelling presents another problem-solving model that the pupils can use, students will only be officially exposed to mathematical modelling at the secondary level, according to the suggestion (MOE, 2020). Instead of diverting to modelling, we advocate a re-look at using real-world problems for problem-solving. In managing the real-world problems, the students are engaged to formulate problems mathematically and check the reasonableness of the solution (MOE, 2020).

The teaching of problem-solving can be split into the three strands of teaching *for* problem-solving (TFPS), teaching *about* problem-solving (TAPS), and teaching *through* problem-solving (TTPS) (Schroeder & Lester, 1989). In TFPS, teaching manifests in the form of concept explanations, skills demonstration by the teacher, and student engagement using mathematical tasks to hone their skills and application of concepts to solve word problems (Leong & Kaur, 2019). TAPS is the explicit teaching and modeling of the language and strategies of problem solving. TTPS sees mathematical problem solving as a pedagogical approach to teach new mathematical content (Toh et al., 2019). This paper focuses on teaching *about* problem solving, emphasizing Check and Expand – Polya's Stage Four. We iterate that problem solving is generally recognised as the missing link in the problem solving instruction in Singapore (Toh, 2020).

Scaffolding

Although educators generally do not encourage excessive teacher involvement in the problem-solving process, Toh et al. (2011a) recognise that some level of scaffolding may be necessary for the pupils to solve problem. Teacher scaffolding should also be tailored to the individual pupils according to their mathematical abilities and needs. Extraneous instructions should not be offered to those able to engage in problem solving without help (Haataja et al., 2019). Instead, the teacher's role should lie in helping pupils monitor their process of thinking in problem-solving. In view of these, Toh et al. (2011a) suggested three hierarchical levels of help that a teacher can provide (Table 2).

Table 2 Levels of Teacher Scaffolding (Toh et al., 2011a)

Level	Feature	Examples of prompts
0	Emphasis on Polya stages and control	What Polya stage are you in now? Do you understand the problem? What exactly are you doing? Why are you doing that?
1	Specific heuristic	Try *looking for a pattern.*
2	Problem specific hints	Why not try smaller numbers? What about sketching a diagram?

According to Wood, Bruner, and Ross (1976) as cited by Haataja et al. (2019), scaffolding can be structured in cognitive, affective, and metacognitive scaffolding. Anghileri (2006) also developed another classification of scaffolding in terms of the levels. Across both structures, there is a prominence in the importance of the cognitive and metacognitive scaffolds.

The researchers distinguish between cognitive and metacognitive scaffolding. Cognitive scaffolding is when the teacher structures the problem for pupils or adapts the problem task to correspond with their competencies (Haataja et al., 2019). With discretion, the teacher aims to support the cognitive activities in the problem-solving process through actions such as explaining and modeling a task, posing questions, and offering hints. On the other hand, metacognitive scaffolding is deliberate teacher action in directing pupils' attention and interaction towards the learning process where the teacher could be advising on the procedures of the problem-solving process. Following this, the nature of the scaffolding by Toh et al. (2011a) is also a blend of cognitive and metacognitive scaffolds, depending on the level of scaffolding the teacher provides. Of the scaffolds categorized by Anghileri (2006), the prominent scaffolds are the metacognitive scaffold of teacher probing and cognitive scaffolds of parallel modeling, simplifying the problem as well as the environmental provisions.

Teacher probing is an example of a metacognitive scaffold where the teacher induces students to explain their thinking through probing questions. Teacher probing can be used to scaffold 'Check and Expand' where the teacher asks probing questions on the critical parts of their workings to the modifications they have made in Stage Four in terms of the reasons for the changes in their workings (if any). The teacher has connected the workings to the modifications and made explicit the students' mathematical understanding. While explaining what they are doing and the purposes for his actions in the solution, the students exercise their metacognitive awareness. As such, they can gain metacognitive insight into their thinking, which may afford opportunities to exercise control in their problem-solving. An example of teacher probing is the Level 0 scaffolding provided by Toh et al. (2011a).

Parallel modeling (Anghileri, 2006) was introduced as a teacher's action in creating and solving a task that shares some of the characteristics to a pupil's problem. This resembles 'Check and Expand', where the teacher takes on the pupils' role of creating and solving an extended problem from the original. This is a form of scaffolding the teacher can take to encourage the making of extensions and generalisations of the problem. While this scaffolding process heavily involves the teacher, a mix of parallel modelling and teacher probing can be used to bridge the resemblance and scaffold the fourth stage of problem-solving. The teacher can ask "What if" questions at the fourth stage of the problem solving and challenge them to solve the problem when some parameters of the problems are changed. Pupils will then be prompted to check if they can apply the mathematics of the original problem as well as compare the mathematics of the original problem and the expanded problem. The teacher has then guided the pupils to think about how they can extend the problem independently by modelling.

Simplifying the problem is another scaffolding suggested by Anghileri (2006). However, this will not be adopted as it is one of Polya's heuristics that the pupils should be independently used to aid them in solving the problem in the first three stages of Polya's model.

Environmental provisions, including the provision of artifacts, are a cognitive scaffold for pupils to make sense of a problem (Anghileri, 2006). This is a form of scaffolding that can be provided for pupils of all abilities since the geometry understanding of most pupils is at the visualization stage of van Hiele levels, and the use of visualization through manipulatives did enhance the pupils' geometry thinking (Tan & Toh, 2013). Since the provision and use of artifacts helped the students understand the problem, it can be perceived as a form of scaffolding useful for Stage One. So, concrete manipulatives can be scaffolding provided for the lower ability pupils when they are solving a geometry problem. However, Anghileri (2006) recognizes that physical manipulatives cannot be provided for all problems and so is less applicable for problem-solving across all mathematical domains. This scaffolding will not be adopted in the exemplar as it is relatively restrictive.

Methodology and Analysis

Research Design

This study aims to propose a design and an enactment of a problem-solving lesson on word problems. The exemplar is an effort towards de-routinizing problems, metacognitively scaffolding students, and encouraging them to look back at their solutions.

The methodological framework is design-based research integrating systematic review as research design. A review of the existing literature on problem solving and studies carried out for secondary school students in Singapore has been made to identify the features that will support authentic problem-solving. The literature was chosen from the ERIC database and Google Scholar using keyword searches. In total, 23 papers were found, and 11 were chosen in the final selection. The literature was selected according to the following selection criteria.

1. The problem-solving methodologies were designed for Singapore student.
2. The problem-solving designs were tested out in the Singapore classroom.

Resources and Targetted Samples

Two other packages for primary school pupils adapted from the key study done by Toh et al. (2011a) have also been studied. The paper proposes how problem solving can be enacted in the mathematics classroom through a synthesis of the existing education literature.

A mathematically rich geometry problem from the 2006 Singapore-Asia Pacific Mathematical Olympiad for Primary Schools was modified in this paper for discussion. A mathematically-rich problem is one that provides the solver opportunities to learn new mathematical concepts and procedures, or develop mathematical processes such as analytical skills, creativity, and metacognition (Yeo, 2007). We illustrate through a detailed commentary proposing how the mathematics classroom teacher can scaffold the pupils to engage in the last stage of the problem-solving. We next discuss why in particular, the topic of geometry was selected for our exemplar reported in this paper.

Methodological Issues: Why Geometry

Geometry is given fewer curriculum hours than word problems and number topics in the Singapore primary mathematics syllabus. The pupils are also less familiar with geometry at the primary level as they are with number topics. In reports and analyses of TIMSS results, there is a smaller difference in achievement across the grades for Geometry (Mullis, 1997). The pupils also underperform in Geometry compared to other mathematical domains (Dindyal, 2006).

That 2D shapes that are taught consistently at every level in the spiral curriculum informs the importance of 2D Euclidean shapes for the pupils' mathematics. However, pupils face many learning difficulties, one of which is in recognizing shapes. According to Clements (2003), primary school pupils typically view congruency as matching the shapes and have a weakness in recognizing and constructing shapes in their non-prototypical representation. This implies that after six years of Geometry instruction, the pupils have not mastered most 2D shapes, especially the concept of congruency. This presents an area of Geometry to contextualize the problem wherein the teacher does not explicitly teach congruency (a topic covered in the secondary syllabus) but uses the problem as an opportunity for pupils to explore the atypical representations of the 2D shapes.

A concept for exploration in Stage Four of the problem-solving model is the relationship between quadrilaterals. For the 2D shape of quadrilaterals at the primary level, pupils are mainly taught on angle and area properties in terms of the area of composite figures. However, they may not have formed the connection between the quadrilaterals, such as the relationship between squares, rectangles and parallelograms. For example, they can check if their solution for a square holds for a rectangle and vice versa.

Furthermore, geometry questions lend themselves towards the formation of alternative solutions more easily (Stupel & Ben-Chaim, 2017). Geometry problems are also easily extendable with how they serve as opportunities for pupils to make mathematical generalizations (Pytlak, 2015), such as 2D shapes from squares to rectangles as mentioned above. Geometry would thus be a suitable strand of mathematics for the introduction to the stage of ‘Check and Expand’ and a successful engagement by the pupils.

Findings and Discussion

Features of an Authentic Problem-solving Package

In response to Research Question 1, “What are the features of an authentic problem-solving package that should be included in this package to focus their problem-solving process on the ‘Checking and Expanding’ phase of their solutions?”, elaboration will be made on how the problem was selected.

Problems considered interesting for pupils to desire to attempt intrinsically motivating and intellectually stimulating (Souviney, 1981). So, the problem chosen had to be interesting to pupils. When pupils first approach a problem, the emotions can influence their interest in the problem (Boekaerts et al., 1995). Suppose pupils perceive a large discrepancy between the perceived problem demands and their own perceived resources to meet these demands, it will elicit negative emotions in them and draw their attention away from the problem. They would rather not attempt the problem or stop at the solution than learn mathematics from the problem. The problem should thus be within the scope of mathematics that the students already have.

In developing this exemplar of problem-solving enactment in the classroom, the following criteria were thus used to select the exemplar problem to be explained in this package.

1. The problems were interesting enough for most, if not all, of the pupils to attempt the problems;
2. The pupils had enough “resources” to solve the problems;
3. The problems were *extendable* and *generalizable; and*
4. The problems had multiple solutions

The exemplary problem was chosen based on these criteria. The problem only requires their knowledge of triangles and squares, but the solutions to the problem are not obvious to the pupils. It will be intellectually stimulating to the pupils without a large gap between the question demands and their resources. As such, the question will be interesting to most pupils to attempt.

Implementing/Enacting Authentic Problem-solving Mathematical Lesson at Primary Level

In response to Research Question 2, “How can authentic problem solving be enacted in a primary school classroom”, in this paper, a mathematically rich geometry problem is used as an exemplar of how the teacher can scaffold pupils to independently find a solution to the problem, as well as extend the problems to find alternative solutions and make generalisations. The commentary included demonstrates the scaffolds teachers can use to induce this last stage of ‘Check and Expand’ from the pupils.

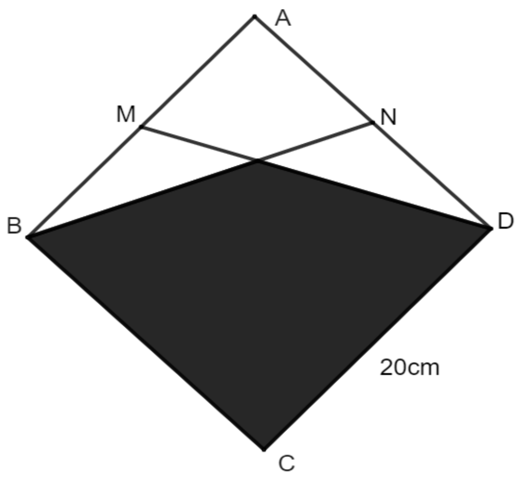
The problem is targeted for pupils at the Primary 5 and 6 levels (students age 11 and 12). The question was chosen to illustrate the big ideas in Mathematics. According to the 2021 Syllabus (MOE, 2020), big ideas express ideas that are central to mathematics and that there is a continuation of the ideas across levels. In this question, the big ideas of Invariance in terms of the properties of the Geometric shapes. The ‘Detailed Lesson Plan’ is attached in Appendix and the ‘Exemplary problem with commentary’ is elaborated in the subsequent section.

In the problem that the researchers selected below, the big idea about proportionality is evident. Proportionality is the relationship between two quantities that allows one quantity to be computed from the other based on multiplicative reasoning (MOE, 2020). As illustrated in the commentary below, the student will explore the big ideas of proportionality as they divide the shapes into subfigures. The presentation taking the form of a commentary obtains inspiration from the lesson package suggested by Toh et al. (2011a). Primary school pupils as ‘text-participants’ have achieved lower fluency and comprehension levels in comparison to the ‘text-analysts’ secondary school pupils. However, there is a significant difference in the language ability levels of the pupils at the primary level and secondary level (Winch et al., 2014). This means that in designing teacher scaffolding, one should be cognizant of simplifying the language for the less fluent primary school pupils. The simplified scaffolding in the form of the “What if” questions, which are not as cognitively loaded, will allow the teacher to continue successfully enacting the lesson and enticing the pupils to solve problem.

Exemplary Problem and Commentary

ABCD is a square of side 20cm. M and N are the midpoints of AB and AD, respectively.
Find the area of the shaded region (Figure 4).

Figure 4 Diagram of square ABCD of side 20cm with points M and N as the midpoints of AB and AD respectively



Suitable hints for Polya Stages I, II

I Understand the problem

(c) Write down the heuristics you used to understand the problem

Teacher scaffold: What is the problem asking for? What information do we need?

Restating the problem – What is the area of the unshaded triangles?
Act it out – What is the relationship between the area of the unshaded triangles?
Draw a diagram – Try dividing up the unshaded triangles into triangles of the same area.

II Devise a plan

(a) Write down the key concepts that might be involved in solving this problem

Area of triangles = ½ base x height

(c) Write out each plan concisely and clearly

Teacher scaffold: What can we do with the divided triangles?

Consider the area of the divided triangle and solve for x (Refer Figure 5). *What is the difference between the area of the square and the sum of the unshaded triangles?*

III Solution(s) to the original question

Solution 1

Label the intersection of BN and DM as O. Draw a line segment AO. The diagram (Figure 5) shows 4 triangles of equal areas since they have the same length and height. Let the area of each triangle in the unshaded part in Figure 5 be *x*.

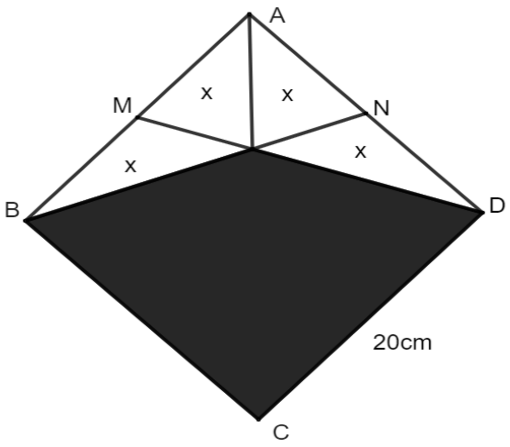
Since M is the midpoint of AB, ΔADM is a quarter of the square.

$$3x = \frac{1}{4} \times 20^2 = 100$$

$$4x = \frac{4}{3} \times 100 = 133\frac{1}{3}$$

Area of shaded part = $20^2 - 133\frac{1}{3} = 266\frac{2}{3}\ cm^2$

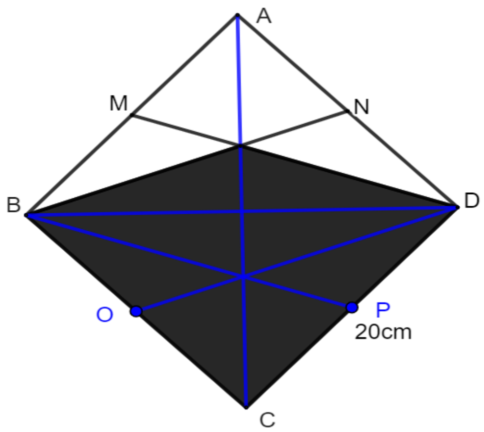
Figure 5 Square ABCD with the four triangles in the unshaded part marked as equal area *x*



Solution 2

The square can be subdivided into 12 triangles with equal areas (Figure 6). O and P are the midpoints of BC and CD respectively.

Figure 6 Square ABCD divided into 12 triangles of equal area



The square is composed of 12 such triangles. The shaded region is made up of 8 such triangles.

So, the area of shaded region = $\frac{8}{12} \times 20^2 = 266\frac{2}{3}\ cm^2$

Suitable hints for Check and Expand (Polya's Fourth Stage)

IV Check and Expand

(a) Write down how you checked your solution

In the above diagram, we use the equation

"Shaded Area + Unshaded Area = Area of whole square"

(b) Write down a sketch of any alternative solution(s) that you can think of

Divide the entire square up into the same triangles in the solution. Find the number of triangles in the shaded area. Take the ratio of the shaded area to the area of the square to find the shaded area.

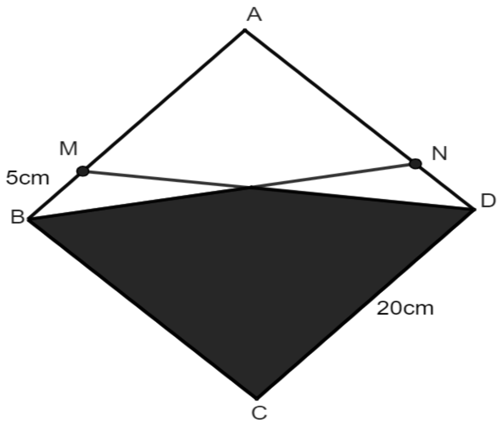
(c) Give at least one adaptation, extension, or generalisation of the problem.

Adaptation

"What if the lengths of BM and DN each is ¼ the side of the square?"

ABCD is a square of length 20cm. The lengths of BM and DN are ¼ the length of one side of the square. Find the area of the shaded region in Figure 7.

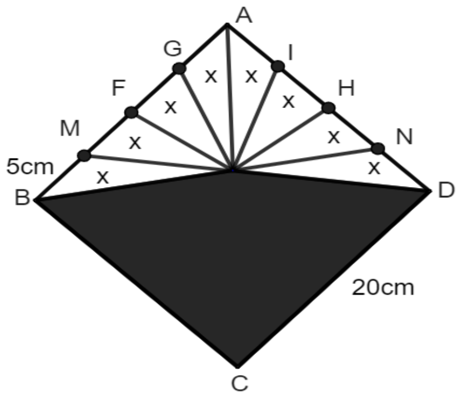
Figure 7 Square ABCD with a different position of the points M and N



Suggested Solution 1 to Adaptation

Label the intersection of BM and DN as O. Label the vertices and draw line segments as shown in Figure 8. The diagram will now show 8 triangles of equal areas since they have the same length and height.

Figure 8 Square ABCD with additional points F, G, H, I, M and N



Since AM is ¾ of the length of the square, the area of ΔADM is $^{3}/_{8}$ of the area of the square.

$$7x = \frac{3}{8} \times 20^2 = 150$$

$$8x = \frac{8}{7} \times 150 = 171\frac{3}{7}$$

Area of shaded part = $20^2 - 171\frac{3}{7} = 228\frac{4}{7}\ cm^2$

How the solution will change

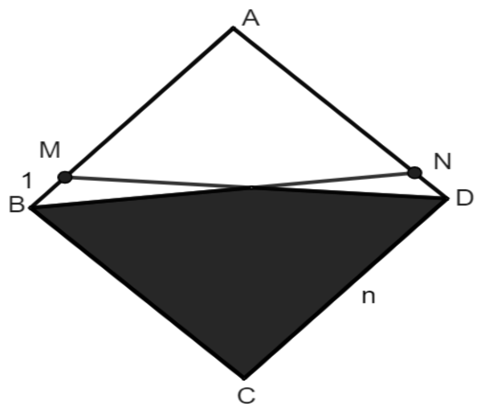
The algorithm remains the same. Only arithmetic changes. When the ratio of length BM to the side of the square halved, the number of unshaded congruent triangles doubled.

Generalization

"What if the length of BM and DN is $\frac{1}{n}$ *the length of the square?"*

*Given that the ratio BM to the side of the square of 1:n for any square, what fraction of the area of the square is the shaded area (*Figure 9*)?*

Figure 9 Square ABCD with length *n* cm and another varied positions of M and N



How the solution will change:

Since BM = 1 cm, we can include n – 1 other points on AB which are equally spaced on AB. Similarly, on AD, we can include n – 1 other points on AD which are equally spaced on AD. Joining all these points to the intersection of BN and DM, we divide the unshaded region in Figure 9 into 2n unshaded triangles of equal areas since they have the same length and height. Let *x* be the area of each of these triangles formed. Then (2n-1) triangles will form ΔADM, which is $\frac{n-1}{2n}$ the area of the square. The algorithm remains the same as the previous parts.

$$(2n-1)x = \frac{1}{2}(n-1)n$$

$$(2n)x = \frac{2n}{2n-1} \times \frac{n-1}{2} \times n = \frac{n-1}{2n-1} \times n^2$$

Area of shaded part =

$$n^2 - \frac{n-1}{2n-1} \times n^2 = \frac{n}{2n-1} \times n^2 cm^2$$

There is an additional final step of comparing the shaded area to the area of the square to get the fraction.

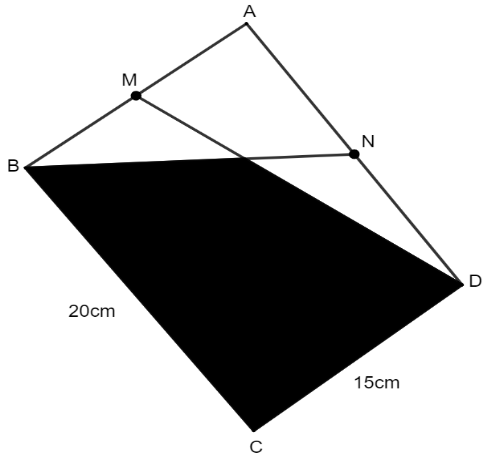
$$\frac{Shaded\ area}{Area\ of\ the\ square} = \frac{\frac{n}{2n-1} \times n^2}{n^2} = \frac{n}{2n-1}$$

Extension 1

"What if ABCD is a rectangle?"

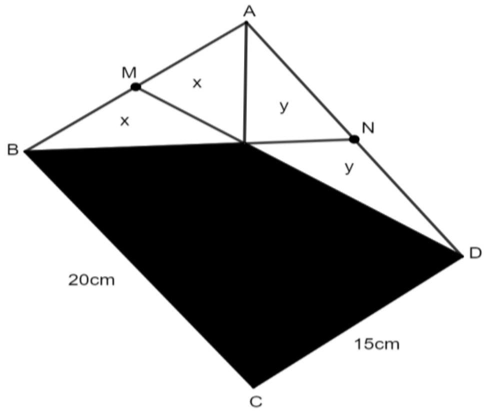
ABCD is a rectangle of sides 15cm and 20cm. M and N are midpoints of AB and AD, respectively. Find the shaded area in Figure 10 or Figure 11.

Figure 10 Rectangle ABCD with M and N as the midpoints of AB and AD respectively



Suggested Solution to Extension 1

Figure 11 Rectangle ABCD with the unshaded parts divided into four triangles



Draw a line segment from A to the intersection of BN and DM, as shown in Figure 11. The diagram will now show 2 triangles of equal area *x* and 2 triangles of equal area *y*.

Since M is the midpoint of AB, the area of triangle ADM is a quarter of the area of the rectangle.

$$2y + x = \frac{1}{4} \times 15 \ \times 20 = 75$$

Similarly, since N is the midpoint of AD, the area of triangle ABN is a quarter of the area of the rectangle.

$$2x + y = 75$$

Solving simultaneously, $x = 25\text{cm}^2$ and $y = 25\text{cm}^2$

$$3x = 75$$

$$4x = \frac{4}{3} \times 75 = 100$$

Area of shaded part = $20 \times 15 - 100 = 200\ cm^2$

How the solution will change

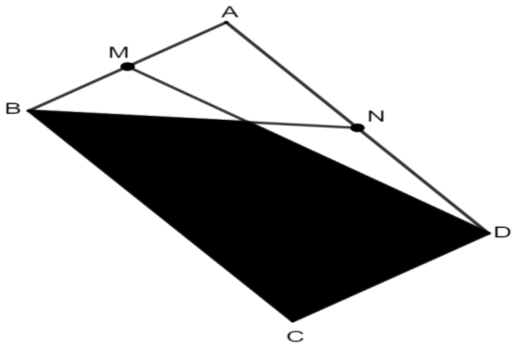
The areas of *x* and *y* are labelled distinctively since that the areas are equal is not obvious. The solver needs to confirm the relationship between the areas of *x* and *y*. Once it is confirmed that the areas are equal, the algorithm of the solution can be applied in terms of the area of the rectangle, with only minor arithmetic changes.

Extension 2

"What if the length and breadth of the rectangle have a specific ratio of 1:2?"

N and M are the midpoints of the length AD and breadth AB, respectively (Figure 12). Given that the ratio of the length and breadth of a rectangle is 1:2, what fraction of the area of the square is the shaded area?

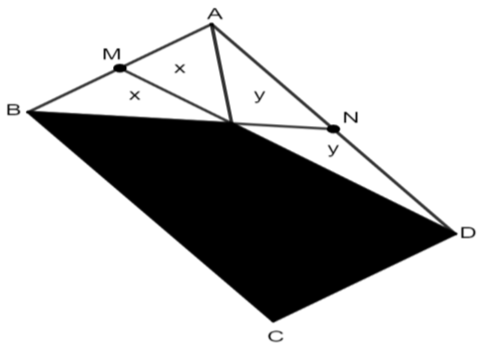
Figure 12 Rectangle ABCD in which M and N are the midpoints of AB and AD respectively, and AD: AB has the ratio 1:2



Suggested Solution to Extension 2

Draw a line segment from A to the intersection of BN and DM, as shown in Figure 13.

Figure 13 Rectangle ABCD with the unshaded part divided into four triangles



The unshaded part of the rectangle has two triangles of equal area *x* and two triangles of equal area *y*. For simplification, let the length be 4 units and the breadth by 2 units, so the length of BM is 1 unit, and AN is 2 units.

Since the breadth is 2 unit and AN is 2 units, then the area of ΔABN is 2 units².

$$2x + y = 2\ units^2$$

Similarly, since the length is 4 units and BM is 1 unit, then the area of ∆ADM is 2 units².

$$x + 2y = 2\ units^2$$

Solving simultaneously, $x = \frac{2}{3}\ units^2$and $y = \frac{2}{3}\ units^2$. The triangles have the same area.

Since AM is ½ of the breadth, the area of ∆ADM is one quarter of the area of the rectangle.

$$2x + y = 2x + x = 3x = \frac{1}{4}\ rectangle$$

$$4x = \frac{4}{3} \times \frac{1}{4}\ rectangle = \frac{1}{3}\ rectangle$$

Area of shaded part = $1 - \frac{1}{3} rectangle = \frac{2}{3} rectangle$

$$\frac{Shaded\ area}{Area\ of\ the\ rectangle} = \frac{2}{3}$$

<u>How the solution will change</u>

Since no value is given beyond the ratio, the solver needs to select lengths useful to simplify the problem.

The areas of *x* and *y* are labelled distinctively since that the areas are equal is not obvious. The solver needs to confirm the relationship between the areas of *x* and *y*. Once it is confirmed that the areas are equal, the algorithm of the solution can be applied in terms of the area of the rectangle, with only minor arithmetic changes.

<u>Mathematics result for the pupils</u>

For any rectangle ABCD, as long as the M and N remain as the midpoint of the respective sides, then the fraction of the shaded area and the unshaded area will be $\frac{2}{3}$ the area of the rectangle. It is apparent that through extending and generalizing the problem, the pupils have the opportunity to discover a new mathematics result. This is aligned to the recent trend of teaching mathematics through problem solving.

Conclusion

Summary and Implications

The importance of critical thinking to students cannot be overstated. Critical thinking enables an individual to think through problems thoroughly, and sorting out relevant information in solving a problem (Stacey, 2007). Critical thinking is important for students, as it prepares them for the challenges that they will face in the lives in the future (Vieira, 2011). Researchers such as Payadnya and Atmaja (2021) believed that "What-If" learning strategy is instrumental in developing students' critical thinking. We believe that teaching students about problem solving, especially the fourth Polya stage on 'Check and Expand', is useful in this aspect.

In this paper, in proposing an authentic problem-solving lesson in the upper primary mathematics classroom, the enactment of the problem-solving phase 'Check and Expand' has been illustrated via an exemplary geometry problem and an accompanying lesson plan. The reader would have noticed that the 'Check and Expand' part of the problem-solving stage makes use of the scaffolding strategies which are clear examples of "What if" strategy. Assimilating this 'Check and Expand' approach of problem-solving as a habit, it is likely that students will be nurtured into an independent problem solver who is not always solving problems, but always seeking for new problems and finding creative solutions. This is an important part of the twenty-first century competency.

Limitations and Future Direction

Although content validation was done by mathematics specialists, the lesson enactment has not been fully trialled in the authentic mathematics classrooms in schools. Nevertheless, we hope that it can motivate teachers to extend their classroom notion of problem solving beyond the use of heuristics to generate a solution towards a metacognitively engaging process, which includes the solver proactively looks back on their solutions.

Significance and Contribution in Line with Philosophy of LSM Journal

This paper presents an enactment of mathematical problem solving in the authentic mathematics classroom based on established mathematical problem solving model. Such an approach has hitherto received little attention and it offers practical guide for classroom teachers with exemplary lesson plan illustrated.

Acknowledgements

The authors would like to acknowledge the funding support from Nanyang Technological University – URECA Undergraduate Research Programme for this research project.

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Appendix: Detailed Lesson Plan

Prior to this lesson, the pupils should have already solved the geometry problem discussed in the paper, and the teacher has already discussed the problem with the pupils up to Stage 3.

<u>Discussion: Check and Expand (15min)</u>

Teacher explains that this stage is called 'Looking Back' in Polya's original model. The teacher will ask pupils what they know about this stage based on its name. The idea behind this stage is that the solver does not stop at a solution to the problem, but the solver should look back at the solution to check and see what he can learn from it.

Teacher explains the reason for the change in name to 'Check and Expand'. We have decided to call it 'Check and Expand' to make some features of looking back and looking forward clearer.

- Check the solution
- Find out if the method can be used to solve other problems
- Pose new problems from the original problem
- Suggest alternative solutions

Teacher introduces the "What If" scaffolding as a strategy that the pupils can use when they 'Check and Expand'. The idea of this scaffolding strategy is that the solver will ask himself "What If" questions to expand the problem by changing parts of the question to see how the solution is affected based on the extensions and generalizations.

Referring to the original problem, the teacher will focus on the different stages by asking them "What If…" questions. The teacher will also ask pupils for their own "What If" questions and write them on the board.

The teacher starts with '**check the solution**'.

<u>Check with pre-conditions:</u> to use the solution to derive a given value
What if the solution of the shaded area is correct? Then what information can you confirm?

The teacher then focuses on '**pose new problems from old**' and outlines the three facets of adapting, extending, and generalizing with examples.

Adapt: to change certain features of the problem
E.g., 1: Changing some numbers
What If the square has a different length?

E.g., 2: Change some conditions
What if the lengths of BM and DN each is ¼ of the side of the square?

Extend: to consider problems that are more 'difficult' or have a greater scope
What if the figure ABCD is a rectangle?

Generalise: to consider problems which would include the given problems as a special example
What if the length of BM and DN is $\frac{1}{n}$ *the length of the square?*

Problem of the day (20min)
In addition to the "What If" questions from the discussion, pupils create their own "What If" questions for the Checking, Adapting, Extending, and Generalisations. These "What If" questions are to be written on the "Mathematics Practical Worksheet". Pupils are to provide solutions to the "What If" questions that were raised up in the discussion and their questions where possible.

Checking and Expanding (20min)

The teacher gets pupils to share their "What If" questions and invites some pupils to present their Stage 4 working on the board. The teacher will ask the pupils to explain his modifications to the question and suggested solutions to the class.

The teacher will use probing questions to elicit the mathematics underpinning their modifications and solution during the pupils' explanation. The teacher will also clarify the new understanding they have gained about their original solutions from engaging in Stage 4.

Closure (5min)

The teacher emphasizes to the pupils that by looking back, they will at least be able to understand the original problem.