Use of Dynamic Geometry Software in the Teaching of Matrix and Transformation: An Exemplar of a Classroom Enactment

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Abstract

Purpose and Research Question - In this paper, we propose a 'Specializing, Conjecturing, Convincing, Generalizing' (SCCG) or SC²G framework for using a dynamic geometry software (DGS) to enact a lesson on "Matrix and Transformation" based on intuitive-experimental approach.

Methodology - A Systematic Literature Review (SLR) focused on the impact of DGS on students' learning, drawing on various learning theories, including Skemp's relational understanding, social dimensional constructivism, and discovery learning.

Findings - We demonstrate with an exemplar the use of SC²G framework in designing one "Matrix and Reflection" lesson for senior high school students.

Significance and Contribution in Line with Philosophy of LSM Journal – An exemplary lesson design of using SC²G framework to facilitate teaching and learning of *Matrix and Transformation* in a Geogebra environment was presented in this paper. This lesson design exemplar provides an example for designing the subsequent lessons on *Matrix and Transformation* based on SC²G framework and hopefully could spur further interest among educators in exploring DGS for mathematics instruction.

Keywords: *Dynamic geometry software; Matrix and transformation; Intuitive-experimental approach; Constructivism; Discovery learning*

Introduction

Matrix and Transformation has been introduced to senior high school curriculum in many countries in the world, for example, in China. In order to help students make adequate preparation for their study of undergraduate mathematics (Ministry of Education [MOE] China, 2003) and equip them with advanced mathematical tools which will facilitate their mathematics learning, basic elements of advanced mathematics have gradually been incorporated into senior high school mathematics curriculum. *Matrix and Transformation* is one such topic that provides students opportunities to engage in deductive reasoning characteristic of advanced mathematics.

The advantages of teaching and learning *Matrix and Transformation* have been advocated by researchers. Hollebrands (2003) affirmed that this topic is not beyond the grasp of senior high school students, and it could positively influence their mathematics learning. The teaching of *Matrix and Transformation* in high school provides students with access to find the connections within mathematics and engages them in critical thinking by invoking various representations (Abdurrahman et al., 2019). Researchers have affirmed that students can transfer what they have acquired from geometrical transformation to algebra and pre-calculus topics they will learn in the future (e.g., Usiskin, 2018).

When introducing new concepts in *Matrix and Transformation*, most textbooks tend to provide definitions without the rationale and their application in the real world. Anecdotal evidence from Chinese classrooms has also shown that teachers excessively emphasize rote memory of special matrices and algorithms when teaching this curriculum portion. Instead of building a meaningful understanding of definitions, students tend to form an "instrumental understanding" (Skemp, 1976) by only memorizing the definitions and the procedures of using them in the problems posed by the teachers or the textbooks.

Curriculum in some countries suggested that teaching and learning geometry at secondary school should adopt an intuitive-experimental approach as recommended by, e.g., the Ministry of Education Singapore (2012). Compared to the traditional instructional approach in Chinese classrooms, an intuitive-experimental approach (Toh & Kaur, 2021) provides the opportunity to shift from teacher-centric to student-centric. Instead of the deductive approach on the one hand and the procedural algorithm approach on the other, the intuitive-experimental approach facilitates students' discovery learning through hands-on activities, thereby providing a seamless transition from algorithmic to rigorous deductive approach in mathematics. The intuitive-experiment approach is usually associated with dynamic geometry software (DGS) environment where students can manipulate objects by just clicking and dragging to explore the mathematical relationships.

This paper proposes a framework [which we call the 'Specializing, Conjecturing, Convincing, Generalizing' (SCCG) or SC^{2}G framework] for designing lessons using the DGS environment within an intuitive-experimental approach. We further present an exemplar of a lesson design to enact a lesson on *Matrix and Transformation*, focusing on geometric reflection, using the framework. By applying DGS during instruction, we aim to deepen students' relational understanding of the transformation matrix, facilitate them to construct the knowledge by themselves through discovery learning and discussion, and show them glimpses of advanced mathematics (related to linear algebra).

Literature review

In conceptualizing our SC^{2}G framework, we have reviewed education literature classified under the categories of (1) Skemp's relational versus instrumental understanding, (2) social dimensional constructivism, and (3) discovery learning.

Relational understanding and instrumental understanding

Skemp (1976) categorized mathematical understanding students acquire into two types: relational understanding and instrumental understanding. Relational understanding means both knowing what to do and why. In contrast, instrumental understanding refers to only knowing the rules without reason. Many researchers have asserted that in emphasizing relational understanding, in addition to acquiring new concepts or procedures, the learners can link their previously held ideas to obtain rich understanding (e.g., Van de Walle et al., 2013).

Studies have also supported Skemp's (1976) claim that relational understanding leads to a more flexible understanding of mathematical concepts and problem-solving procedures. Students who have developed a relational understanding of mathematics are better at adapting the methods to new tasks and applying mathematical knowledge creatively (e.g., Reason, 2003). Skemp (1976) pointed out that there were many experienced teachers whose understanding of mathematics belongs to the realm of instrumental understanding. This could lead to an instructional method that might overemphasize the rules and algorithms used in problem-solving procedures while ignoring the underlying rationale. Studies have also shown that compared to solving problems which require relational understanding, preservice teachers performed better in solving problems which only require instrumental understanding (e.g., Patkin & Plaksin, 2019).

Constructivist learning

Piaget proposed the concept of constructivism based on the belief that acquiring knowledge is an ongoing process of self-construction (Driscoll, 2005). He argued that knowledge is invented and reinvented through learners' interaction with the surrounding environment (Piaget, 1964). During the process of knowledge construction, learners build their own understanding of the world by using their preexisting mental schemes to make sense of the new experiences (Bodner, 1986).

Another constructivist researcher Vygotsky proposed the sociocultural theory of development, which emphasizes that social interaction and cultural factors are central to shaping an individual's learning and development. (Vygotsky, 1930; DeVries, 2000). According to Vygotsky's sociocultural constructivism, cognitive functions are products of social interaction within a community and cultural context (Topçiu & Myftiu, 2015). Communication and language, as cultural tools, play a fundamental role in facilitating learners' knowledge construction (Vygotsky, 1965). Specifically, communication is considered a key component of developing relational understanding in mathematics learning (Hiebert, 1997). Students co-construct an understanding of culturally established mathematical practices by sharing their reasoning and listening to others' thinking process. (Steele, 2001).

Active learning and discovery learning

The role of an instructor changes from a source of knowledge to a facilitator of learning under the constructivism, instead of an imparter of knowledge such as a traditional teacher or lecturer (Bauersfeld, 1995). A facilitator helps learners to build their own knowledge, while a teacher gives didactic lectures that transmit subject matters to students (Adom et al., 2016). By self construction and co-construction, students transfer their role from passive listeners and receivers to active learners. Bruner views students as active problem-solvers and thinkers (Akpan & Kennedy, 2020), making sense of the world through collaboration and discussion (Bruner, 1997). He argued that the goal of education is not merely to develop students' subject knowledge but also to cultivate them as autonomous and self-propelled thinkers who possess the passion and ability to learn by themselves after formal schooling (Bruner, 1961). In order to achieve this goal, Bruner proposed the discovery learning approach, which encourages students to actively engage with the learning materials while the teacher provides the necessary scaffolding. Active involvement not only makes students pay more attention to their learning, but also impulses them to construct a deeper processing of information (Svinicki, 1998). Through active participation, students will gradually develop a sense of ownership over their own learning (Cattaneo, 2017). When students take the discovery method to approach learning, they tend to carry out the learning activities autonomously. They will also more likely be intrinsically motivated by the act of discovery itself (Bruner, 1961).

DGS and Geogebra

Through DGS, students' mode of learning has evolved from paper and pencil to a simple click of a computer mouse. In this way, students have the opportunity to create and manipulate geometrical objects by simply clicking and dragging for infinitely many possible cases, thereby being empowered to explore geometric relationships, produce geometric conjectures, and make conjectures and generalizations (Bakırcı et al., 2011; Oldknow, 1997).

Consisting of both graphic window and algebra window, which can be displayed simultaneously, Geogebra is a DGS system that allows dual modes of operation. Users can either operate the geometric tools with a mouse to construct geometrical objects in the graphic window or directly key in algebraic functions and commands in the area of the algebra window (Doğan & Içel, 2011). While the visual representation of all objects is presented in the graphics window, their algebraic representation is shown in the algebra window simultaneously. In Geogebra, these two windows remain synchronized with each other. In other words, any changes made to the graphical objects in the graphics window will result in corresponding changes in their algebraic and numerical representation in the algebra window, and vice versa.

Studies have also shown that the use of DGS has positive effects on students' mathematics learning. Research indicated that Geogebra not only facilitates students in constructing new knowledge, but also help them connect it with their prior knowledge (e.g. Shadaan & Leong, 2013), consistent with Piaget's view of constructivist learning. Uygun (2020) reported that using Geogebra could enhance students' conceptual understanding and reasoning ability of geometric transformation. Hollebrands (2003) showed that with the help of Geogebra, students' understanding of transformation shifted from simple motions towards functions and mappings. We believe that this shift of understanding also plays a crucial role when students learn "Matrix and Transformation". Yao and Manouchehri (2019) have found that DGS can effectively support students to construct mathematical generalizations about geometrical transformations. By allowing learners to interact with mathematical objects in a dynamic and visually engaging environment, DGS enables learners to not only discover new properties but also validate their conjectures and observations.

Studies have shown that compared to students using the traditional chalk-and-talk method, students in a Geogebra environment had significantly higher academic achievements (Bekene Bedada & Machaba, 2022; Doğan & Içel, 2011). Specifically, students taught using Geogebra better mastered conceptual knowledge, procedural knowledge and problem solving skills (Alkhateeb & Al-Duwairi, 2019). With the assistance of Geogebra, students performed better in associating different representations of trigonometric functions in terms of algebraic and graphic views. It was also found that the use of Geogebra improved students' mathematical communication skills, allowing them to express their mathematical understandings and ideas better (Kusumah et al., 2020). Furthermore, the benefits of Geogebra appear to be long-lasting, as students who used it continued to outperform their counterparts even after several months (Birgin & Topuz, 2021). These findings suggest that Geogebra is effective not only for improving student mathematics achievement but also for enhancing their knowledge retention.

DGS also has a positive influence on the students' attitude towards mathematics. Turk and Akyuz (2016) found that students perceived using DGS, such as Geogebra, as less tedious and more enjoyable than drawing on paper and pencil, which often required repeated sketches. Birgin and Topuz (2021) also showed that compared to students receiving traditional instruction, students exposed to DGS generally showed a more positive attitude towards the mathematics class. GeoGebra's visual and dynamic nature enables it to capture students' attention, arouse their curiosity, and motivate them to participate in more interesting activities. Hosseini et al. (2022) found that students showed greater enthusiasm and interest in the subject martials, when hands-on activities based on DGS were used during mathematics class. According to Owusu et al. (2023), Students indicated a clear preference for using Geogebra in mathematics classroom as they found that Geogebra made the learning process more exciting and funnier. During a Geogebra-assisted lesson, students were more engaged and actively interacted with both the software and their peers (Nzaramyimana, 2021). After learning in a Geogebra environment, students perceived that Geogebra is a valuable tool which could reduce their cognitive load and encourage a more creative learning environment (Yimer & Feza, 2019). Students in the DGS setting also had a higher level of self-efficacy, showing more confidence about their problem-solving ability (Isiksal & Askar, 2005).

The ultimate goal of geometry is to induct students into deductive reasoning, which poses many difficulties for students. By providing students with visual aid and an interactive manipulative environment, DGS can bridge the gap between geometrical objectives and deduction (Jones, 2000). In fact, studies have shown that dynamic computer environments such as DGS can inspire students to link their intuitive notions with formal aspects of mathematics knowledge (e.g., Sutherland et al., 2004). Intuition is the first step before engaging in formal mathematical proofs. According to Bruner (1960), a leap of thought is formed step by step, and intuition plays an important role before rigorous deduction emerges. Intuition, in other word, guessing, is less rigorous than proof but more iconic or visual and more towards the whole problem than to particular parts. By intuitive thinking, one can grasp the structure of the problem without explicit analysis (Bruner, 1979).

Method

In this study, we conceptualize a framework of teaching *Matrix and Transformation* and provided an exemplar of the lesson using the framework developed through critical synthesis of scientific evidence to answer the aforementioned specific research topic by reviewing previous studies based on an educational Systematic Literature Review (SLR) technique (Guillaume, 2019; Purssel & McCrae, 2020).

A total of 39 sources, including book chapters and journal articles, were reviewed. The sources were found via a systematic search on Google Scholar and the Education Resources Information Center (ERIC) database.

SLR was conducted in the following areas with the number of sources stated:

(1) Relational versus instrumental understanding – four sources (Skemp, 1976; Reason, 2003; Van de Walle et al., 2013; Patkin & Plaksin, 2019).

(2) Constructivist learning – nine sources (Vygotsky, 1930; Piaget, 1964; Vygotsky, 1965; Bodner, 1986; Hiebert, 1997; DeVries, 2000; Steele, 2001; Driscoll, 2005; Topçiu & Myftiu, 2015).

(3) Active learner and discovery learning – seven sources (Bruner, 1961; Bauersfeld, 1995;

Bruner, 1997; Svinicki, 1998; Adom et al., 2016; Cattaneo, 2017; Akpan & Kennedy, 2020).

(4) DGS and Geogebra – 13 sources (Bruner, 1960; Bruner, 1979; Jones, 2000; Hollebrands, 2003; Sutherland et al., 2004; Isiksal & Askar, 2005; Doğan & Içel, 2011; Shadaan & Leong, 2013; Turk & Akyuz, 2016; Alkhateeb & Al-Duwairi, 2019; Yao & Manouchehri, 2019; Yimer & Feza, 2019; Kusumah et al., 2020; Uygun, 2020; Birgin & Topuz, 2021; Nzaramyimana, 2021; Bekene Bedada & Machaba, 2022; Hosseini et al., 2022; Owusu et al., 2023).

Conceptualization of a lesson

Pre-requisite. This lesson presumes that students have mastered lower secondary knowledge of reflection in mathematics, including identifying the refection relationship between two figures, and conducting reflection of geometrical figures over a point or a line in the 2-dimensional cartesian coordinate system. In addition, we assume that the students have also gained related prior knowledge from physics, such as properties of reflection of light. In the prior lessons on matrix, we further assume that students have already acquired the skills of performing matrix multiplication and, in particular, calculating the post-multiplication of a 2×2 matrix by a 2×1 vector. Nevertheless, aligned to the existing curriculum documents, we assume that the knowledge of matrix multiplication remains at the computational level, without geometric interpretation.

Thinking and working mathematically. When learning new mathematics content, students must be cognitively engaged in doing and thinking mathematically (Holton & Thomas, 2021). To begin with specializing, one could interpret thinking mathematically, which refers to considering particular cases of a more general situation in the mathematics question. It ends with *generalizing* when students can find the underlying pattern of all the specific cases and apply it to a much broader class. When getting stuck between *specializing* and *generalizing*, students will likely need more methods to attack the problem, such as *conjecturing*, *justifying* and *convincing*, *distilling* and *mulling*, and finding *hidden assumptions* (Mason et al., 2010). Conjecturing and justifying convincingly are two of the fundamental methods of attack. While conjecturing is more of an individual, convincing includes, in addition to oneself, communication and discussion with others. According to Mason et al. (2010), there are three stages of convincing: convince oneself, convince a friend and convince a sceptic. In alignment to learn through discovery and discussion, in this paper, we select specializing, conjecturing, convincing and generalizing to construct our framework to design our exemplar lesson plan on teaching matrix and transformation.

SC^2 G framework of lesson enactment

Based on structure of thinking mathematically (Mason et al., 2010), we propose the 'Specializing, Conjecturing, Convincing, Generalizing' (SCCG) or SC^{2}G framework, which consists of four stages that we suggest to involve in the teaching of matrix and transformation.
S: Specializing
C: Conjecturing
C: Convincing
G: Generalizing

Specializing some examples as hands-on activities. At the beginning, teachers introduce the connection between matrices and transformation in geometry. In particular, the geometrical interpretation of post-multiplying a 2×2 matrix by a 2×1 vector. The introduction is for activating students' prior knowledge about matrix and transformation. In the constructivist learning theory, prior knowledge is essential as it serves as a foundation for constructing new knowledge (Bada & Olusegun, 2015). To help the learners incorporate the new knowledge into their construct, teachers must elicit their relevant prior knowledge (Baviskar et al., 2009).

The lesson begins with some concrete examples to enable students to key in special points in Geogebra. Students are asked to observe the transformation of the points they multiplied by the special matrices $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$. This approach aligns with the constructivist view, which emphasizes that teachers should allow students to have some *direct experience* with subject matters (Inan & Inan, 2015). By allowing students to engage in hands-on activities, teachers can facilitate them to form some sense of the geometric representation of matrix multiplication.

Conjecturing. In DGS, students are provided with the environment to click and drag to observe or discover the geometrical property (Toh & Kaur, 2021). As in any DGS, students can drag the point to different positions. By directing the students to objects and images of transformation by given matrices, students are led to conjecture the relation between the original point and its image. The dragging function of the DGS not only enables students to change the input vector more directly but also helps them to abstract general rules by noting the features that remain invariant under the drag mode (Dienes, 1967; Hollebrands, 2003). In trialing with many different sets of values of x and y, and dragging the end points to change the resultant vectors, students can gradually discover that the feature that remains invariant among all the changes, which corresponds to reflection over *x*-axis, *y*-axis, *y*=*x* and *y*=-*x*.

The Geogebra environment provides students the freedom to conduct self-discovery learning, exploring the relationship between the original points and their images by themselves. This self-driven process emphasizes the role of students as active knowledge constructors and fosters their autonomy in the learning process (Bruner, 1961).

Convincing. In the convincing stage, students are provided with invariant points, unlike the specializing stage, where points do not lie on lines of reflection. For example, when considering the reflections over the *x*-axis, students are asked to plot images of the specially selected points (7,0), (2,0), (4,0), (5,0), (-3,0), (-2,0), (-8,0) and (-7,0), observe and attempt to explain to their classmates why the points stay invariant. Similarly, when considering the reflections over the *y*-axis, students are asked to plot images of the specially selected points (0,3), (0,5), (0,2), (0,7), (0, -3), (0,-5), (0,-2) and (0, -4), and explain the invariance of the images. These activities activate students' prior knowledge of reflection and allow the teachers to check whether their students have made the correct conjecture and whether they make sense of the geometrical interpretation of the matrices and the effects of transformation.

Generalizing and finding transformation matrices. After the first three stages, students could infer that reflection about a line (passing through the origin) on the two-dimensional cartesian coordinate system is represented by certain matrices (This corresponds to the bigger idea of the linear transformation in Linear Algebra). The teacher could next guide their students to deliberate how the matrix associated with each reflection transformation can be determined.

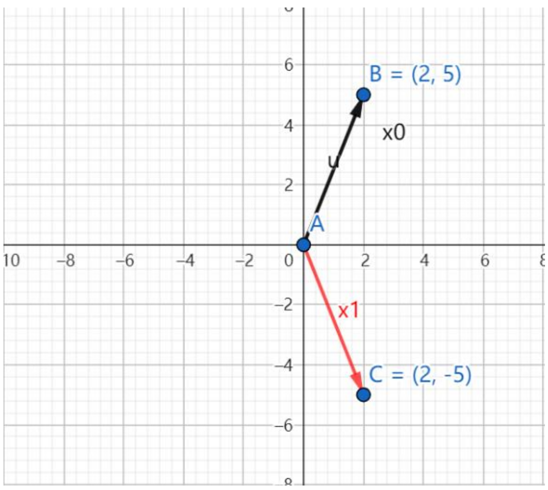
In the generalizing section, teachers lead the students to consider the images of two "special" points (1,0) and (0,1) (These points are the standard basis of the two-dimensional cartesian plane) and to find the possible relations between the images and the transformation matrices. By self-discovery and group discussion, we aim that students can abstract that the reflections of (1,0) and (0,1) construct the first column and second column of the corresponding transformation matrices. The discussion here provides the opportunity for students to communicate their reasoning process with their partners, as well as check their findings.

An example of the worksheet is provided below:

Worksheet

1) Find the answers of the following matrix multiplication by keying in the points in Geogebra https://www.geogebra.org/m/pccsbpsc, and mark the points and their images in the Cartesian coordinate system in Geogebra (Figure 1).

Figure 1 Cartesian coordinate system



① $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1.6\\-2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}4\\-2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-5\\2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-8\\-3\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}3\\-3\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

② Drag the point B to different positions and observe its images. Based on your exploration, suggest the relation between the point B and its images.

My answer: ____

③ Plot the points (7,0), (2,0), (4,0), (5,0), (-3,0), (-2,0), (-8,0) and (-7,0) in Geogebra and find their images. What are their images? Can you explain why their images are like that?

④ Plot (0,1) and (1,0) in Geogebra and find their images. What are their images? Based on your exploration, suggest the relation between the images of (1,0), (0,1) and the corresponding transformation matrix $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.

My answer: ____

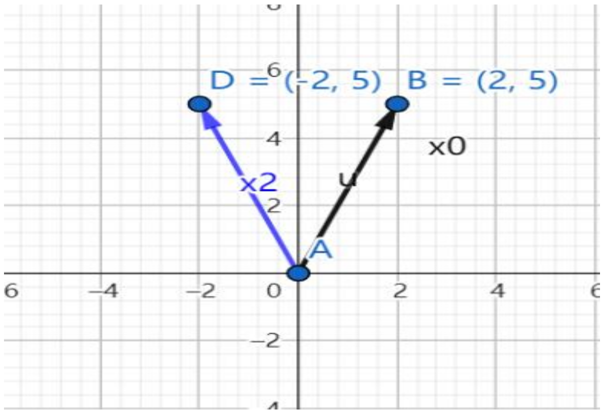
2) Find the answers of the following matrix multiplication by keying in the points in Geogebra https://www.geogebra.org/m/rasnv7jp, and mark the points and their images (Figure 2) in the Cartesian coordinate system in Geogebra.

① $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}-1.6\\-2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}4\\-2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

Figure 2 Points and images

$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}4\\ 4\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-5\\ 2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-8\\ -3\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}3\\ -3\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$



② Drag the point B to different positions and observe its images. Based on your exploration, suggest the relation between the point B and its images.

My answer: ____

③ Plot the points (0,3), (0,5), (0,2), (0,7), (0, -3), (0, -5), (0, -2) and (0, -4) in Geogebra and find their images. What are their images? Can you explain why their images are like that?

④ Plot (0,1) and (1,0) in Geogebra and find their images. What are their images? Based on your exploration, suggest the relation between the images of (1,0), (0,1) and the corresponding transformation matrix $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$.

My answer: ____

3) Find the answers of the following matrices multiplication. Verify your results (Figure 3) with Geogebra, and mark the points in the Cartesian coordinate system

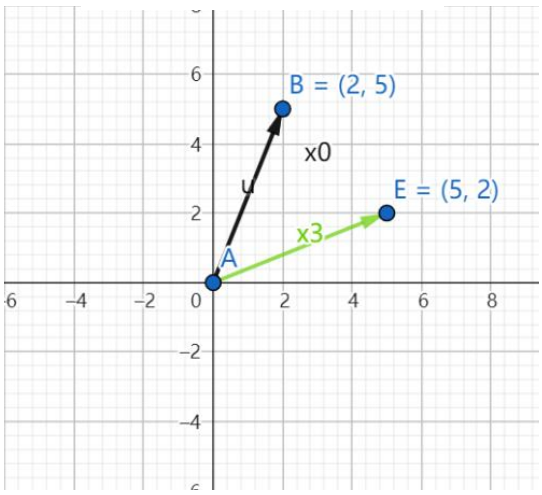
① $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2\\ 5\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1\\ 3\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1.6\\ -2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}4\\ -2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}6\\ 4\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-5\\ 2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-8\\ -3\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$ $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix}=\begin{pmatrix}\ \end{pmatrix}$

Figure 3 Verifying results



② Drag the point B to different positions and observe its images. Based on your exploration, suggest the relation between the point B and its images.

My answer: ____

③ Plot the points (2, 2), (-3, -3), (5, 5), (-4, -4), (1,1), (-6, -6), (7,7) and (-1, -1) in Geogebra and find their images. What are their images? Can you explain why their images are like that?

④ Plot (0,1) and (1,0) in Geogebra and find their images. What are their images? Based on your exploration, suggest the relation between the images of (1,0), (0,1) and the corresponding transformation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

My answer: ____

4) Find the answers of the following matrices multiplication. Verify your results with Geogebra, and mark the points (Figure 4) in the Cartesian coordinate system.

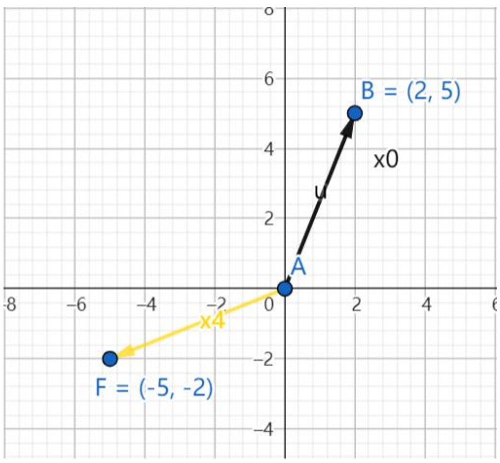
① $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = (\ )$ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = (\ )$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1.6 \\ -2 \end{pmatrix} = (\ )$ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = (\ )$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = (\ )$ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -5 \\ 2 \end{pmatrix} = (\ )$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -8 \\ -3 \end{pmatrix} = (\ )$ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = (\ )$

Figure 4 Marking the points



② Drag the point B to different positions and observe its images. Based on your exploration, suggest the relation between the point B and its images.

My answer: ____

③ Plot the points (2, -2), (-3, 3), (5, -5), (-4, 4), (-1,1), (6, -6), (-7,7) and (-15,15) in Geogebra and find their images. What are their images? Can you explain why their images are like that?

④ Plot (0,1) and (1,0) in Geogebra and find their images. What are their images? Based on your exploration, suggest the relation between the images of (1,0), (0,1) and the corresponding transformation matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

My answer: ____

5) Based on your previous answers, suggest a relationship between the reflection images of (1,0) and (0,1) over x-axis, y-axis, $y = x$, $y = -x$ and the corresponding reflection matrices, and discuss with your partners in the group.

My answer: ____

Conclusion

We have presented the SC^{2}G framework for teaching "matrix and reflection" at senior high schools by riding on the affordance of technology. This is particularly useful for countries which might not have rode on the affordance of technology for teaching geometry in their curriculum. Our proposed framework is based on the literature review conducted in mathematics education using DGS during instruction, built on various traditional learning theories.

The SC^2G framework is an attempt to design a lesson using the intuitive-experimental approach when teaching geometry. What is illustrated in this paper is a section on reflection in *Matrix and Transformation*. We hope that students can construct their own knowledge through self-discovery and group discussion in the environment of Geogebra. This lesson design not only aims to facilitate students with their current learning of matrix and reflection, but also prepare them for learning linear algebra in the future, particularly the prior knowledge of linear transformation and the role of a basis of a vector space in linear transformation.

The SC^2G framework has also been used to design the remaining lessons on *Matrix and Transformation* and will be trialed on high school students to study its efficacy. The result of our subsequent studies will be reported sometime in the future. Although to date the lesson has yet to be trailed in an authentic mathematics classroom, we hope that our conceptualization will inspire an interest into exploring the use of DGS in teaching geometry-related topics in high schools.

While the design is specially developed for the Chinese educational context, the literature review and the learning theories we used transcend the nation's boundary. We believe this study's design is equally applicable in other countries. Nonetheless, further research is necessary to explore the effectiveness of this design in various cultural settings.

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